Numerical Study of the Entrance Flow and its Transition in a Circular Pipe

By
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円管内助走流の遷移の研究
神田英貞

概要
円管流の層流安定に関する問題を、実験結果と数値計算から、特に遷移距離と助走距離に着目して研究する。これまで、この課題は流体力学の主要な問題であるにもかかわらず明確に定義されていない。遷移距離をベルマウスのある場合とない場合について実験的に定める。遷移点は色素除流脈が最初に乱れ始まる点とする。また、助走区間の速度、圧力分布の数値解を、レイノルズ数が10000の場合について、2次元軸対称流れ方程式からガウス・セイデル法を反復法により求め、このようにして、レイノルズ数が大きくてもヘンゲー＝ポワシル流れが発生することを示し、さらに助走距離を求めて、遷移が助走区内で発現することを確かめる。同じ計算スキームを用いて、ベルマウスのある場合の流入条件をモデル化するために入口の中心に近い点、および壁近傍の点に軸対称円環状の有限の乱れを与え、層流から乱流への遷移をシミュレートする。数値的に得られる遷移点は壁近傍の流れ（限界流）が最初に乱流すると定め、実験値と比較し、良好に一致をみた。さらに、レイノルズ数が無限大の流れでは、"有限機械乱れに対して不安定な主流の速度分布は必ず変曲点を持つ"、及び"主流が不安定である場合、機械乱れの位相速度は主流の最大速度を超えない"というレイリーによる命題を検討する。

Summary: The experimental data and results of prior investigations lead to defining the problem of the transition from laminar to turbulent flow in a circular pipe. So far, the subject has been a major problem for hydro- and aeromechanics, and yet it seems not to have been clearly defined. Therefore, the flow field of the circular pipe is examined with particular emphasis on the entrance and transition length. The transition lengths were experimentally determined for two inlet shapes without and with bellmouth. The transition point is the starting point of oscillation of color dye filament in the pipe. Two-dimensional, time-dependent computational schemes have been devised for determining the flow development and the corresponding pressure drop in the entrance region at Reynolds numbers (Re) based on the pipe diameter of 10, 100, 2000, and 10000. The stream-function vorticity formulation and the Poisson equation were applied, using the Gauss-Seidel iteration method. The method, the pressure drop, and the convective and the viscous terms are compared with the experimental results and also with the previous analysis. Thus, it is numerically verified that the laminar solution for Hagen-Poiseuille flow exists regardless of the Reynolds number and experimentally observed that the real transition length is much shorter than the entrance length. Moreover, finite and axisymmetric disturbances were superimposed on points near the inlet and wall of the pipe. The transition point for numerical analysis is the point where the limiting stream line changes to back flow on the wall. It was, for the first time, found that the transition length is predicted fairly satisfactorily by the computational simulation. Finally, two Lord Rayleigh's theorems are discussed in comparison with the results of the simulation.
### Notation

<table>
<thead>
<tr>
<th>Symbol</th>
<th>Definition</th>
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<tbody>
<tr>
<td>Am</td>
<td>amplitude constant for stream function</td>
</tr>
<tr>
<td>Ct</td>
<td>coefficient for time increment</td>
</tr>
<tr>
<td>D</td>
<td>pipe diameter</td>
</tr>
<tr>
<td>dr</td>
<td>radial space increment of mesh system = $\Delta r$</td>
</tr>
<tr>
<td>dt</td>
<td>time increment = $\Delta t$</td>
</tr>
<tr>
<td>dz</td>
<td>axial space increment of mesh system = $\Delta z$</td>
</tr>
<tr>
<td>DZR</td>
<td>aspect ratio = $dz/dr$ ($=\Delta z/\Delta r$)</td>
</tr>
<tr>
<td>i</td>
<td>axial point of mesh system. $i=1$: inlet, $i=10$: outlet</td>
</tr>
<tr>
<td>j</td>
<td>radial point of mesh system. $j=1$: centerline, $j=10$: wall</td>
</tr>
<tr>
<td>Lep</td>
<td>dimensionless entrance length = $zep/(D*Re)$</td>
</tr>
<tr>
<td>Let</td>
<td>dimensionless transition length = $zet/(D*Re)$</td>
</tr>
<tr>
<td>m</td>
<td>iteration number</td>
</tr>
<tr>
<td>n</td>
<td>number of time step</td>
</tr>
<tr>
<td>p</td>
<td>pressure</td>
</tr>
<tr>
<td>Pc ($z^*, r$)</td>
<td>point in cylindrical coordinate</td>
</tr>
<tr>
<td>Pm ($i, j$)</td>
<td>point in mesh system</td>
</tr>
<tr>
<td>Q</td>
<td>total flux</td>
</tr>
<tr>
<td>r</td>
<td>radial coordinate</td>
</tr>
<tr>
<td>R</td>
<td>pipe radius = $D/2$</td>
</tr>
<tr>
<td>Re</td>
<td>Reynolds number based on pipe diameter</td>
</tr>
<tr>
<td>Rer</td>
<td>Reynolds number based on pipe radius</td>
</tr>
<tr>
<td>t</td>
<td>dimensionless time = $t/(R/uo)$</td>
</tr>
<tr>
<td>u</td>
<td>axial velocity component</td>
</tr>
<tr>
<td>uo</td>
<td>mean axial velocity component</td>
</tr>
<tr>
<td>v</td>
<td>radial velocity component</td>
</tr>
<tr>
<td>z</td>
<td>axial coordinate</td>
</tr>
<tr>
<td>zep</td>
<td>entrance length</td>
</tr>
<tr>
<td>zet</td>
<td>transition length</td>
</tr>
<tr>
<td>$z^*$</td>
<td>dimensionless axial length = $z/(D*Re)$</td>
</tr>
<tr>
<td>$z^{**}$</td>
<td>dimensionless axial length = $z/D$</td>
</tr>
<tr>
<td>$\Gamma$</td>
<td>circulation per unit length</td>
</tr>
<tr>
<td>$\Delta r$</td>
<td>radial space increment of mesh system = $dr$</td>
</tr>
<tr>
<td>$\Delta t$</td>
<td>time increment = $dt$</td>
</tr>
<tr>
<td>$\Delta z$</td>
<td>axial space increment of mesh system = $dz$</td>
</tr>
<tr>
<td>$v$</td>
<td>kinematic viscosity</td>
</tr>
<tr>
<td>$\psi$</td>
<td>stream function</td>
</tr>
<tr>
<td>$\omega$</td>
<td>vorticity</td>
</tr>
<tr>
<td>*</td>
<td>multiplication operator</td>
</tr>
<tr>
<td>**</td>
<td>square operator</td>
</tr>
<tr>
<td>/</td>
<td>division operator</td>
</tr>
</tbody>
</table>
Entrance Flow and its Transition in a Circular Pipe

1. INTRODUCTION

1.1 Summary of Previous Work

The stability of pipe flows with constant, circular cross section has one of the most interesting problems, and many researchers have studied it since Hagen (1839) and Poiseuille (1841), however, it still remains in the question. It is one of the principal problems of modern hydrodynamics. This problem is important not only for engineering application of pipe flow, but also for the extension of our fundamental knowledge on turbulent flow. The mechanism of turbulence, particularly the transition from laminar to turbulent flow still remains unsolved.

There are two types of flow in the circular pipe. If the fluid velocity in the pipe is low enough, the flow is laminar. Then, at a sufficient distance downstream from the inlet of the pipe, the velocity vector at any point is parallel to the centerline and the well-known parabolic velocity profile of Hagen-Poiseuille flow is established. If the velocity is increased beyond some critical value, there will be a sudden change in the resistance of the flow. Turbulence sets in, and along with the appearance of radial and azimuthal velocity components, we find that all three components of the velocity are randomly fluctuating in time. However, even in the case of turbulent flow, if the water is steady in a container and drawn into the pipe without any disturbance, the laminar flow continues near the inlet and it changes suddenly into the turbulent flow some distance downstream. The relation between the resistance encountered by, and the velocity of, water moving through the pipe, presents itself mostly in one or other of two simple forms. The resistance for the turbulent flow is generally proportional to the square of the velocity, and for the laminar flow it takes a simple form and is proportional to the velocity.

Hagen [27] observed several times that the transition from the turbulent flow to the laminar one depends on the radius of the pipe, on the velocity, and on the temperature, or the viscosity of the water. The turbulent flow will become laminar as soon as any one of the three quantities mentioned, or all three of them together, decrease below a certain limit.

Reynolds [29] observed, during his experiments of determining the exact circumstances under which the change of law took place, that there were two critical values for the velocity in the pipe, the one at which eddies or turbulence began if the water were approximately steady when drawn into the pipe, the other at which the water became stable and eddies changed into steady motion for any disturbance, as the water proceeded along the pipe. The latter critical value was called 'the real critical value' by him and was much less than the former at which it would become unstable for infinitely small disturbances. For the former critical value, Reynolds undertook experiments by means of color bands in glass pipes in 1880. The water was drawn through the pipes out of a large glass tank, in which the pipes were immersed, arrangements being made so that a streak or streakes of highly colored water entered the pipes with the clear water. He obtained the following general results and determined the exact circumstances under which the change of law of resistance took place.
1) When the velocities were sufficiently low, the streak of color extended in a
beautiful straight line through the pipes as the laminar flow in Fig. 1.
2) If the water in the tank had not quite settled to rest, at sufficiently low
velocities, the streak would shift about the tube, but there was no appearance of
disturbance.
3) As the velocity was increased by small stages, at some point in the pipe, always
at a considerable distance from the bellmouth or intake, the color band would all at
once mix up with the surrounding water, and fill the rest of the pipes with a mass of
colored water as the turbulent flow in Fig. 1. Any increase in the velocity caused the
point of break down to approach the bellmouth, but with no velocities that were tried
did it reach this.
4) The law of the critical point, at which the transition from laminar to turbulent
flow occurs, is completely expressed by using the Reynolds number Re, which is
defined as follows:

\[ Re = \frac{D \cdot u_0}{v} \]

where D is the pipe diameter, uo the mean axial velocity component, and v the
kinematic viscosity.

In general, the transition from laminar to turbulent flow can be understood through
the theory of hydrodynamic stability. In this theory, a time independent laminar
velocity profile is chosen. A small velocity perturbation is introduced into the flow and
it is assumed that the laminar profile is not changed by the presence of the
perturbation. Then a determination is made as to whether the disturbance grows,
decays, or remains the same size as a function of time. The many results of analytical
attempts to calculate a critical Reynolds number for the parabolic profile in the pipe
have all concluded that the pipe flow is stable to all infinitesimal disturbances. Previous
theoretical analyses and experiments are briefly summarized as follows:

1) The flow of a viscous incompressible fluid in a circular pipe is linearly stable to
both axisymmetric and non-axisymmetric infinitesimal disturbances.
2) This theoretical conclusion is supported by experimental evidence, provided
that the experiments are carried out with considerable care as regards the shape of the
inlet of the pipe and the inlet flow conditions.
3) However, in most of experiments performed with a pipe with a sharp inlet edge, the flow usually becomes unstable when the Reynolds number exceeds a value of about 2000.

4) Therefore, the Hagen-Poiseuille flow is probably unstable to small but finite disturbances. The explanation of the observed transition to turbulence in this flow requires finite-amplitude instabilities /24/.

So far, the transition length, which is the distance between the inlet of the pipe and the point where the transition from laminar to turbulent flow occurs, has been seldom studied in detail. The transition length is one of the typical characteristics of the transition. Consequently, it is very important to determine the transition length.

On the other hand, the laminar flow in the pipe can be spatially divided into two regions. The fluid velocity will be very nearly uniform across the inlet. The velocity profile will then evolve into the parabolic profile with increasing distance downstream as shown in Fig. 2. The region from the inlet to where the parabolic profile is established is called the entrance region and its length is the entrance length; the other may be called the fully developed region. The flow of parabolic velocity is well known as pipe Poiseuille or Hagen-Poiseuille flow. Many researchers have made efforts to determine the entrance length and found that it is a direct function of the Reynolds number and for a Reynolds number of 2000 it is approximately 100 to 200 pipe diameters long. These results will be discussed in 1.3. The entrance transition problem is that of determining the velocity profile in the entrance region of the pipe as a function of distance from the inlet. This problem is different from that of determining the transition length.

Most of the theoretical efforts have been expended in studying the stability for the fully developed flow, but some efforts were made for the flow in the entrance region and the results showed the instability, although they are much larger than the real critical value. These results will be discussed in 1.4.

Accordingly, as to the investigation on the instability of a circular pipe, its flow field and conditions will be conveniently classified as follows:

1) The entrance region under infinitesimal disturbances.
2) The entrance region under finite or large disturbances.
3) The fully developed region under infinitesimal disturbances.
4) The fully developed region under finite or large disturbances.

Experimental results show the occurrence of the transition in the cases of 1, 2, and 4. Moreover, investigations on the problem are categorized by four approaches: experimental approach, stability analysis, theoretical analysis, and direct numerical
Table 1. Research categories

<table>
<thead>
<tr>
<th>Approach</th>
<th>Entrance region</th>
<th>Fully devel. region</th>
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<tbody>
<tr>
<td><strong>Experiment</strong></td>
<td></td>
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</tr>
<tr>
<td>Arakawa, Matsunobu</td>
<td>o</td>
<td>Leite</td>
</tr>
<tr>
<td>Hagen</td>
<td>o</td>
<td>Leite</td>
</tr>
<tr>
<td>Kirsten</td>
<td>o</td>
<td>Stettler, Hussain</td>
</tr>
<tr>
<td>Nikuradse</td>
<td>o</td>
<td></td>
</tr>
<tr>
<td>Oshima, Kanda</td>
<td>o</td>
<td></td>
</tr>
<tr>
<td>Poiseuille</td>
<td>o</td>
<td></td>
</tr>
<tr>
<td>Ramanprian, Tu</td>
<td>o</td>
<td></td>
</tr>
<tr>
<td>Reynolds</td>
<td>o</td>
<td></td>
</tr>
<tr>
<td>Schiller</td>
<td>o</td>
<td></td>
</tr>
<tr>
<td><strong>Theoretical analysis</strong></td>
<td>o</td>
<td>o</td>
</tr>
<tr>
<td>Boussinesq</td>
<td>x</td>
<td>Davey, Drazin</td>
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<tr>
<td>Langhaar</td>
<td>x</td>
<td></td>
</tr>
<tr>
<td>Mohanty, Asthana</td>
<td>x</td>
<td>Garg, Rouleau</td>
</tr>
<tr>
<td>Schiller</td>
<td>x</td>
<td>Gill</td>
</tr>
<tr>
<td>Sparrow, Lin</td>
<td>x</td>
<td>Salwen, Grosch</td>
</tr>
<tr>
<td><strong>Stability analysis</strong></td>
<td>o</td>
<td>o</td>
</tr>
<tr>
<td>Huang, Chen</td>
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<tr>
<td>Kuwabara S.</td>
<td></td>
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<tr>
<td>Tatsumi</td>
<td></td>
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<tr>
<td><strong>Numerical simulation</strong></td>
<td>o</td>
<td>x</td>
</tr>
<tr>
<td>Dixon, Hellums</td>
<td></td>
<td>Kyrazis</td>
</tr>
<tr>
<td>Kanda, Oshima</td>
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<tr>
<td>Kanda, Oshima</td>
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<td>Kawamura</td>
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<td>Koyari</td>
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<tr>
<td>Vrentas</td>
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</table>

Simulation as shown in Table 1. Here, research efforts are classified into two simple groups according to whether they concentrated on the entrance region or on the fully developed region. In Table 1, symbols “o” and “x” indicate, respectively, whether the transition took place or not. In short, we can easily find that most of the symbols “o” belong to the “entrance region” group of Table 1. Thus, the transition takes place very often in the entrance region.

Reynolds' observations have been, unfortunately, quite neglected until recently: “Under no circumstances would the disturbance occur nearer to the trumpet than about 30 diameters in any of the pipes, and the flashes generally, but not always, commenced at about this distance”.

1.2 Objectives

This research work has both experimental and computational aspects for determining the transition and the entrance length.

The transition length, which is the distance between the inlet of the circular pipe and the point where the transition from laminar to turbulent flow occurs, was experimentally determined. The entrance length, which is the distance between the inlet and the point where the velocity profile grows to the fully developed, parabolic...
distribution, was computationally found and they were compared with each other.

The following studies were carried out:

1) The transition lengths were experimentally determined for two inlet shapes without and with bellmouth. The transition point for experiment is the starting point of oscillation of color-dye filament or color band in the pipe flow.

2) The entrance lengths at Reynolds numbers of 10, 100, 2000, and 10000 were numerically calculated.

3) The transition lengths are compared with the entrance lengths.

4) The transitional flow was numerically simulated for the case of the inlet without bellmouth. Some disturbances given just at the inlet are investigated whether they decay or grow. The transition point for numerical analysis is the point where the limiting stream line changes to back flow, that is, the first separation takes place and back flow appears on the wall. Finally, the transition lengths were numerically obtained at Reynolds numbers of 2700 and 10000.

The transition lengths from observation and numerical analysis might disagree since it is not accurately verified that the first separation point causes the starting of oscillation of color-dye filament at the same axial distance from the inlet. We assumed that both transition lengths from experiment and numerical analysis agree with each other for the following reason. In the entrance region, the flow is hydrodynamically developing and is, therefore, essentially of the boundary-layer type near the wall. Thus, instability of developing flow can be expected to occur near the wall at sufficiently large Reynolds numbers.

Various models for describing turbulence have developed in last 100 years. The idea of numerically simulating the transition length is new. Any turbulent model was not attempted. It is concluded that the transition length depends on the velocity distribution at the inlet, which relates to the shape of the inlet and it takes place only within the entrance region.

The experimental results of Leite [21] showed that two dimensional disturbances are less stable than three dimensional ones. Leite also found that the non-symmetric part of the disturbance decayed more rapidly than the symmetric part. Moreover, Squire’s theorem [5] says that it is sufficient to consider only two-dimensional disturbances to obtain the critical Reynolds number for a viscous fluid. They yield a lower limit for the critical Reynolds number. Hence, the two-dimensional, full Navier-Stokes equations are used based on the stream-function vorticity formulation and the following assumptions are made.

1) The fluid is incompressible and Newtonian fluid with constant viscosity and density.

2) The flow is axisymmetric.

3) The gravity and the other external forces are neglected.

1.3 Entrance Length

Through theoretical analysis and computational simulation, the entrance length, velocity distributions, and pressure drops are numerically calculated for the flow in the entrance region. In this region, a larger pressure drop per unit length than that in the
fully developed region is required since velocities are accelerated from uniform at the inlet to the parabolic profile. Moreover, a boundary layer are formed and develops downstream in the entrance region.

The entrance length ze is usually expressed in a dimensionless form as Eq. (1) by dividing itself by the diameter of the pipe and the Reynolds number,

\[ \text{Lep} = \frac{\text{ze}}{\text{D} \times \text{Re}} \]  

(1)

where Lep is the dimensionless entrance length. A subscript “p” means parabolic. A dimensionless axial length from the inlet \( z^* \) is expressed as Eq. (2) in the same dimensionless form of Eq. (1),

\[ z^* = \frac{z}{\text{D} \times \text{Re}} \]  

(2)

At the entrance length, \( z^* \) is equal to Lep. Moreover, Lep is denoted as Eq. (3), which shows that Lep is the entrance length per flux.

\[ \text{Lep} = \frac{\text{ze}}{\text{D} \times \text{Re}} = \frac{\text{ze}}{\text{D}^2 \times \text{u}_0 / \nu} = \frac{\text{ze}}{4 \text{Q} / \nu} = \frac{\pi \nu \text{ze}}{4 \text{Q}} \]  

(3)

where, \( \text{u}_0 \) is the mean axial velocity, \( \nu \) is the kinematic viscosity and \( \text{Q} \) denotes the total flux across a section: \( \text{Q} = \pi \times \text{u}_0 \times \text{D}^2 / 2 / 4 \). Lep was calculated 0.1 for Reynolds numbers of above 50 [11].

Boussinesq [27] was the first to investigate theoretically the flow field in the region and obtained the dimensionless entrance length of 99% of the fully developed value: \( \text{Lep} = 0.26 / 4 = 0.0625 \). Boussinesq’s value \( \text{Lep} = 0.065 \) seems to be in agreement with the Nikuradse’s experimental data taken at some distance away from the inlet. Schiller [31] calculated it by assuming that velocity profiles are constant near the central core and parabolic near the wall: \( \text{Lep} = 0.115 / 4 = 0.0288 \). The velocity profiles by Schiller show good agreement with Nikuradse’s data for about a third of the initial length from the inlet [32]. Langhaar [20] obtained \( \text{Lep} = 0.0575 \) for 99% of the fully developed value by using the Bessel function. The analysis of Mohanty and Asthana [23] presented \( \text{Lep} = 0.075 \) for 99.9% of the fully developed value by dividing the entrance region into two parts. Boundary-Layer approximation is applied to the first part, and therefore the value is considered reasonable above Re of 500.

Recently, computers can be used to numerically simulate time-dependent flow phenomena and to create detailed pictures of flow fields. By a computational numerical approach, the solution of the complete Navier-Stokes equations can be obtained for given initial and boundary conditions, although the three-dimensional solution is somewhat limited. For the problem, it is often assumed that a uniform velocity exists at the inlet of a pipe. However, Vrentas, Duda, and Bargeron [41] assumed that the developing velocity field in the entrance region will be significantly influenced by the velocity field in the upstream region. The upstream conditions had a considerable influence on the velocity field in the entrance region below a Reynolds number of 50, but no influence above a Reynolds number of 150. The entrance length
for 99% of the fully developed value is obtained by an iterative method: \( Lep=0.33, 0.047, 0.048, 0.0535, \) and 0.0562 at Reynolds numbers of 1, 50, 150, 250, and under Boundary-Layer approximation, respectively. Kyrazis [19] defined arbitrarily the entrance length as the length from the inlet to a point where the maximum radial velocity component is less than one percent of the mean flow velocity. Then the numerical results shows that \( Lep \) is 0.04 at a Reynolds number of 50. Moreover, Kyrazis showed that the assumption of parallel flow for the linear stability theory is invalid in the portion of the entrance region in which stability calculations have been made. The radial velocity is as high as 19% of the mean flow at \( z^* = 0.004 \) and, even as far downstream as \( z^* = 0.01 \), the radial velocity reaches a value of 12% of the mean flow. Koyari [16] considered the entrance length as a distance from the inlet to a point where the radial pressure gradient vanishes \((-dp/dr=0)\) and calculated \( Lep \) is 0.0325. The velocity grows to 98% at the downstream end \((z^* = 0.136)\). Kanda and Oshima [11] found the similarity of the velocity distributions for the flow above \( Re \) of 50: \( Lep=0.045, 0.055, \) and 0.1 for 98%, 99%, and 100%, respectively. Their results are presented in detail in 3.4. In addition, Leite observed, compared with the calculated results of Boussinesq, that satisfactory agreement existed at the low Reynolds numbers; however, at high Reynolds numbers, the formula gave values to be too large for the entrance length. For instance, at \( Re=13000 \), the required length for a satisfactory profile was only 0.8 times the predicted length: \( Lep=0.065*0.8=0.052 \).

The brief results of investigations are summarized in Table 2, where \( p \) denotes the pressure and \( v \) the radial velocity.

### 1.4 Transition Length

We define three types of Reynolds numbers: a transitional Reynolds number, a
critical Reynolds number, and the minimum critical Reynolds number. If the speed of the flow along a circular pipe increases, transformation of some individual disturbances into turbulence occurs. With a greater increase in the velocity, the transition takes place more violently. A "transitional Reynolds number" is a Reynolds number at which a transition takes place, and has a wide range of values under the same inlet and experimental conditions. The minimum value of transitional Reynolds numbers is defined as the "critical Reynolds number" under some condition. The critical Reynolds number depends very strongly on the conditions which prevail in the inlet of a pipe, such as the shape of bellmouth, as well as in the approach of it. Schiller [27] found that in order to obtain a high critical Reynolds number, it is especially important to round off the inlet of the pipe. As far as is known, there is no upper critical Reynolds number. Ekman [32] reached a value of up to 40000. The minimum value of critical Reynolds numbers is defined as the "minimum critical Reynolds number", which is approximately from 2000 to 2300. Below this value, the flow remains laminar under infinitesimal disturbances. As stated in 1.1, Reynolds found, below the minimum critical Reynolds number, that the water became stable and eddies changed into steady motion for any disturbance imbedded at the inlet, as the water proceeded along the pipe. He called the minimum critical Reynolds number "the real critical value".

Experiments for determining the transition length have rarely been performed. Instead, most experiments have been carried out in order to observe conditions under which the transition occurs and to determine a critical Reynolds number and the minimum critical Reynolds number. Unfortunately, the experiments have often been performed with brass pipes. The transition takes place in a pipe, so the transition length must be shorter than the pipe length or the distance from the inlet to the measuring point. Of course, the transition does not occur just at the downstream end. Accordingly, we regard the pipe length, or the distance above mentioned, as the transition length when this is not measured.

Transition length < pipe length

or

Transition length < distance from inlet to measuring point

So far, the transition length zet is often presented in a dimensionless length divided only by the diameter of a pipe. In order to compare the entrance length to the transition length, the same dimensionless unit is strongly desirable as Eq. (4).

\[
\text{Let} = \frac{\text{zet}}{D \cdot \text{Re}}
\]

(4)

where Let is the dimensionless transition length shown in Fig. 1. A subscript "t" means turbulent.

The first investigation was done by Hagen, and he used three brass tubes of 0.281,
0.405, 0.596 cm in diameter and 47.6, 108.7, 104.3 cm in length, respectively. We cannot precisely estimate the transition length due to the pipes being of brass. However, the dimensionless transition length can be roughly calculated as less than 0.0734 at Reynolds number of 2300 and less than 0.0241 at Reynolds number of 7000: Let<47.6/(0.281*2300)=0.0736, 47.6/(0.281*7000)=0.0241.

Reynolds showed, by dimensional analysis, that the transition depends on the dimensionless expression, that is, the Reynolds number. He conducted experiments 29 times with color bands to obtain the critical velocities at which steady motion breaks down. Reynolds numbers of the experiment vary within 11500 and 14400. The average value is 12900, which is a critical Reynolds number and not the minimum critical Reynolds number. He observed that the transition would never occur in the entrance region at less than 30 diameters in any of his pipes. The dimensionless transition length by Reynolds is about 0.00233: Let=30/12900=0.00233. It is important to confirm whether the transition length of Reynolds’ experiment is necessarily reproducible or not. He also carried out experiments to determine the minimum critical Reynolds number by means of resistance in the pipes and it was approximately 2030.

Schiller [31] performed his experiments by measuring the pressure drop of a flow of water through smooth cylindrical brass pipes. He showed that the least necessary transition length in order to create an abrupt breakdown of the laminar flow must be not less than 100–130 pipe diameters. He observed the abrupt change in the pressure drop at a transition length of 0.039 by using the pipe with sharp edges. Later, Schiller [8] conducted experiments by inserting a thin thread of dye into the fluid (see 4.2).

According to the measurements performed by Kirsten [32] the transition length is 50 to 100 diameter. Leite conducted experiments in a Lucite pipe of 1.25 in. in diameter and 73 ft. (700 diameters) long. High-pressure air (90 lb./(in. **2)) was used. The peripheral distributions of amplitude of disturbances were measured at points after the disturbance generator which was mainly placed in the fully developed region. The small disturbances were decayed after 12.4 diameters downstream of sleeve at Re=13000: 12.4/13000=0.00095. He suggested the singificant conclusions (see 4.4). Arakawa and Matsunobu [11], [2] found that turbulent flows occurred at a Reynolds number of 2000 with straight pipes of 1 cm diameter and 17.5 and 35 cm in length. The dimensionless transition length is less than 0.00875: 17.5/2000=0.00875. Ramapriyan and Tu [28] used a copper tube of 5 cm internal diameter and 880 cm in length. The test section is followed by another copper tube 30 cm long closed at the downstream end and distributions of turbulent velocities were obtained at a Reynolds number of 2870. Let is less than 0.0613: Let<880/(5*2870)=0.0613. They also observed fully turbulent flow at all times in the case of Re=2100 and Let is below 0.0838. Moreover, no turbulent plug was observed for Re of below 2000 by Stettler and Hussain [35], and random puff was recorded at Re=2100 and z/D=330, where z is the pipe length and its diameter D is 2.54 cm. Let is below 0.157. Oshima and Kanda [12], [13], [14] carried out color-dye experiments to measure the transition length under two different inlet shapes without and with bellmouth and these results are presented in detail in the following section.
A vortex breakdown, which occurs in a swirling pipe flow, is different form of the transition to turbulent flow. It is somewhat valuable to mention the transition length of a vortex breakdown. We define it as follows:

Transition length = distance from inlet to tip of a bubble

The tip of a bubble means an axial station of beginning of breakdown. Uchida and Nakamura [38] investigated a stable axisymmetric vortex breakdown and a subsequent spiral instability in a swirling air flow with fewer disturbances. They measured it enough to represent a stagnation region using a pipe of 8 cm in diameter and 100 cm in length, and the Laser Doppler Velocimeter. The transition length was 15 cm at Re=2300: Let=15/(8*2300)=0.000815. They [39] also made an experiment, using color-dye filament, on a spiral type of vortex breakdown for a swirling water flow in a pipe of 5 cm in diameter. The transition length was about 12 cm at Re=2100: Let=12/(5*2100)=0.00114.

The results concerning the transition length are summarized in Table 3.

### Table 3. Summary of transition length

<table>
<thead>
<tr>
<th>Authors</th>
<th>Re</th>
<th>Transi. length</th>
<th>Bell Mouth</th>
<th>Notes</th>
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<tr>
<td></td>
<td></td>
<td>(cm)</td>
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<tr>
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<tr>
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<td>3500</td>
<td>0.05&gt;</td>
<td>(17.5)</td>
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<td>Hagen</td>
<td>7000</td>
<td>0.241&gt;</td>
<td>47.6</td>
<td>no</td>
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<tr>
<td>Kirsten</td>
<td>13000</td>
<td>(50-100) * D</td>
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<td></td>
</tr>
<tr>
<td>Leite</td>
<td>2702</td>
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<td>8.7 * x</td>
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<td>Pfenninger</td>
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<td></td>
<td></td>
<td></td>
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<td>838</td>
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<td>100</td>
<td>D=8, vortex breakdown</td>
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<tr>
<td>Uchida</td>
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<td>D=5, vortex breakdown</td>
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<td>axisym./non-axisym.</td>
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<td></td>
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<tr>
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</tr>
<tr>
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<td>Salwen, Tatsumi</td>
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<td>axisymmetric</td>
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</table>

1.5 Critical Reynolds Number by Stability Analysis

The critical Reynolds numbers according to theory and from observation often
widely disagree. The theoretical critical Reynolds number indicates the point on the wall at which amplification of some disturbances begins. The disturbances flow downstream and the observed point of transition will be downstream of theoretical value. In other words, the experimental critical Reynolds number exceeds its theoretical value. In this paper, however, for reasons of simplicity, neither critical Reynolds number is not distinguished accurately.

Small-disturbance stability theory has been applied several times to Poiseuille flow. Sexl began the theory of the instability of pipe flows for axisymmetric disturbances. Analytical studies have shown that the pipe flow in the fully developed region is linearly stable to small axisymmetric disturbances and also to small non-axisymmetric disturbances [3], [6], [7], [30]. It is generally accepted that the flow is unstable only if nonlinear amplitude-dependent forces can exert a significant influence on the behavior of a small but finite disturbance. Hence a theoretical study of nonlinear stability properties is called for. To date, however, only two major theories of nonlinear stability have been developed [33]. The first concerns weakly nonlinear effects and the second concerns nonlinear critical layers. Both theories are promising, but not determinant in deciding conclusively the nonlinear stability of Hagen-Poiseuille flow. However, Tatsumi [37], by using the boundary-layer approximation and linear stability, calculated numerically a minimum critical Reynolds number for developing laminar flow in the entrance region of the circular pipe: 19400 at the point 8.5 times the pipe diameter downstream from the inlet \( \text{LET}=8.5/19400=0.000438 \). Huang and Chen, by linear stability due to axisymmetric [10] and due to non-axisymmetric disturbances [9], also obtained minimum critical Reynolds numbers 39800 at \( \text{LET}=32/39800=0.0008 \) and 39560 at \( \text{LET}=0.00123 \), respectively. These values are much less than the dimensionless entrance length of 0.1. Kuwabara [18] obtained a critical Reynolds number of 1213 for the fully developed flow by the nonlinear hydrodynamical stability. However, since a mean flow is strongly deformed, his velocity profile appears to belong to that in the entrance region. The brief results of these investigations are summarized in Table 3.

1.6 Numerical Analysis

Solutions of the basic equations in fluid dynamics have been sought for since the formulation of the Navier-Stokes equations. It became possible to numerically solve the basic equations of motion for certain difficult problems with the development of computers. The principal merit of direct simulation is to take account of the effect of viscosity. Tani [36] provided some comments about the effect of viscosity on the problem of instability, which were introduced by Reynolds and Prandtl: the viscous fluid is unstable, under which circumstances the cause of instability is the viscosity, although the effect of viscosity is also in the direction of stability.

The references on computational, numerical analyses of hydrodynamic stability are fewer than expected. One of the difficulties of numerical simulation is obtaining the correct, time-dependent solutions for flows at high Reynolds numbers. The accuracy of numerical results should always be confirmed. Moreover, it is only with considerable difficulty that oscillations of turbulent flow should be distinguished from
those of a numerical instability. Dixon and Hellums [4] changed the amplitude of disturbances and estimated the relationship of a critical Reynolds number and the magnitude of amplitude. The appearance of distortions in wave form of the vorticity fluctuation are reproducible and give a reasonably well-defined Reynolds number and amplitude relationship. The results show both the strong amplitude dependence of stability and a minimum critical Reynolds number of about 2000. The flow field length is 18.5 times diameter of a pipe. The dimensionless transition length is less than 0.037, 0.00617, 0.00185, 0.000185 at Reynolds numbers of 500, 3000, 10000, and 100000, respectively. Kyrazis constructed a numerical model of Leite’s experiment at a Reynolds number of 13000, utilizing the full, nonlinear, time-dependent Navier-Stokes equations. The calculated disturbance magnitude varied over a 100000:1 range for fully developed Hagen-Poiseuille flow. He found that a large disturbance is stable and propagate downstream as if it were an infinitesimal disturbance and that undisturbed Hagen-Poiseuille flow is stable for both small and large disturbances. The first result is against the observation of Leite. Patera and Orszag [24] studied the stability of pipe flow to axisymmetric disturbances by direct numerical simulation of the incompressible Navier-Stokes equations. There is no evidence of finite-amplitude equilibria at any of the wavenumber/Reynolds number combinations invested, with all perturbations decaying on a time scale much shorter than the diffusive (viscous) time scale. In particular, decay is obtained where amplitude expansion perturbation techniques predict equilibria, indicating that these methods are not valid away from the neutral curve of linear stability theory. Starting with the experimental data of Laufer, Kawamura [15] simulated directly the flow field for the circular pipe by using a three-dimensional and explicit scheme. The results of the velocity distributions agree well with the results of Laufer. However, the transition length is ignored.

2. Experiment of Transition

The experimental apparatus, a water channel, a circular pipe and bellmouth are shown in Figs. 3 and 4, which are placed in Oshima laboratory of the Institute of Space and Astronautical Science. After water in a container appears to be steady, fluid enters the smooth circular pipe of 3 cm in diameter and 300 cm in length from the large container. The bellmouth and pipe are made of acrylic resin and the diameter of the inlet of the bellmouth is 8.7 cm and its central length is 4 cm. At the downstream end of the pipe there is a plate with several types of holes which control the amount of flow. The measured data were for 858≤Re≤40530. The Reynolds number was calculated from the total flux:

\[ \text{Re} = \frac{4 \cdot Q}{\pi \cdot D \cdot v} \]

The experiments were carried out for two cases: (a) without bellmouth and (b) with the bellmouth. The main objectives of the experiments are to measure the transition length by the color-dye method and to determine conditions under which the transition occurs. The color-dye filament or color band flows downstream in the
**Entrance Flow and its Transition in a Circular Pipe**

![Water channel diagram](image)

**Fig. 3.** Water channel.

![Circular pipe and bellmouth](image)

**Fig. 4.** Circular pipe and bellmouth.

The central part of the cross section of the pipe, not near the wall. The change of color-dye filament in the pipe was observed by four persons (K. Oshima, Y. Oshima, H. Kanda, and Y. Ishii). It was difficult to determine the precise transition point because the starting point of oscillation moves a considerable-distance upstream and downstream, and is therefore not clearly distinguishable. We assumed that the transition length is the distance from the inlet to the point where the color-dye filament begins oscillation perpendicular to the main flow. Johannesen and Lowe [40] repeated Reynolds' dye experiment with the same apparatus which has survived at the University of Manchester. They showed that a filament of colored water does not mix all at once with the surrounding water. There is an intermittent part between laminar and turbulent regions.

Table 4 shows the experimental data of the transition length. The importance of reproducibility was well ascertained for the experiment without bellmouth, but not the one for with the bellmouth.

In the case without bellmouth, the transition length is about 19 cm for Re=2702 (Let=0.0023) and 7 cm for Re=3766 (Let=0.00062). Below Re=1930, no transition appears even under manual vibration of the container. In the case with the bellmouth, the transition length is about 170 cm for Re=7000 (let=0.0081), between 80 and 120 for Re=11500 (Let=0.0023–0.0035) and 70 for Re=15440 (Let=0.0015). Below Re=5790, no transition appears even under manual vibration of the container. No special disturbances were used to make the transition occur. The results are plotted in Fig. 19 of the following section.

Both zet and Let become small as the Reynolds number increases.
Table 4. Experimental transition length

<table>
<thead>
<tr>
<th>Re</th>
<th>Transition Length(cm)</th>
<th>Let</th>
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<tr>
<td>1245</td>
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<td>1930</td>
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<td>7</td>
<td>0.00061</td>
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<tr>
<td>1061</td>
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<td>1206</td>
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<td>1351</td>
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<td>1673</td>
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<td>2027</td>
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<td>2059</td>
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3. Calculation of Flow Field in Entrance Region

3.1 Mesh System and Basic Equations

The use of finite differences is not without limitation. Because of limited speed and main storage of the computer, only two-dimensional flow could be studied through the time-dependent and iterative scheme.

The basic equations are written by the finite-discrete formulation on the discrete points formed by the mesh system. Because of the azimuthal symmetry of a circular pipe, it is utilized the rectangular grid system composed of the region $0 \leq z \leq z_0$ and $0 \leq r \leq 0.5$, where $z$ and $r$ are the axial and radial coordinates, respectively, and are
made dimensionless by dividing by the diameter of the pipe. The rectangular mesh system used is schematically shown in Fig. 5. Each point in the pipe is expressed as \( P_c(z^*, r) \) in the cylindrical coordinates or as \( P_m(i, j) \) in the mesh system. The relations between \( P_c \) and \( P_m \) are:

\[
  z^* = (i-1)\Delta z, \quad r = (j-1)\Delta r,
\]

where \( \Delta r = 0.5/20 = 0.025 \). The axial point \( i \) takes an integer value between 1 (inlet) and 10 (outlet) and the radial point \( j \) is between 1 (center line) and 10 (wall). The number of the mesh points becomes \( 10 \times 10 \), in order to obtain the accurate numerical solutions, it is necessary to make the number of mesh points, 10 and 10, large. On the other hand, there are some severe limitations on computer main storage size and computational power. Since the accuracy of the finite difference scheme and the computation time are somewhat proportional to each other, a compromise as to the grid spacing, time step, and accuracy of the iterations must be made. Dixon used the grid spacing and time step of \( \Delta z = 0.175 \), \( \Delta r = 0.1 \), and \( \Delta t = 0.15 \). Kawamura used the three-dimensional mesh system in which the number of mesh points is 32, 64, and 32 in the radial, azimuthal and axial coordinates, respectively. The aspect ratio of \( \Delta r \) to \( \Delta z \) was 1.

For the entrance region described above we consider the two-dimensional, unsteady flow of an incompressible Newtonian fluid with constant viscosity and density. We neglect gravity and external forces. Consequently, the dimensionless forms of the stream-function vorticity equation and the Poisson equation are written in the cylindrical coordinates for axisymmetric flow.

**Stream-function vorticity equation:**

\[
  \frac{\partial \omega}{\partial t} + \frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial \omega}{\partial r} \right) + \frac{\partial}{\partial z} \left( r \frac{\partial \omega}{\partial z} \right) + \frac{\omega}{r} \frac{\partial}{\partial r} \left( r \frac{\partial \omega}{\partial r} \right) = \frac{1}{\text{Re}} \left( \frac{\partial}{\partial r} \left( r \frac{\partial \omega}{\partial r} \right) + \frac{\partial^2 \omega}{\partial z^2} \right)
\]

**Poisson equation:**

\[
  -\omega = \frac{\partial}{\partial r} \left( \frac{1}{r} \frac{\partial \psi}{\partial r} \right) + \frac{\partial^2}{\partial z^2} \left( \frac{\psi}{r} \right)
\]

Table 5 gives the initial and boundary conditions, where \( I_1 = 10 - 1 \) and \( I_2 = 10 - 2 \). The whole fluid particles start moving downstream with uniform velocity in the pipe at...
the initial time. The no-slip boundary condition is on the wall and any vorticity does not exist at the inlet and on the center line of the circular pipe. Moreover, the outlet condition is given by extrapolation.

The axial and radial velocity, u and v, are calculated from the derivatives of stream function in the general method:

\[ u = \frac{1}{r} \frac{\partial \psi}{\partial r} \]  \hspace{1cm} (7) \\
\[ v = -\frac{1}{r} \frac{\partial \psi}{\partial z} \]  \hspace{1cm} (8)

The pressure drop within the region is written in the Poisson form.

\[ \frac{\partial}{\partial r} \left( r \frac{\partial p}{\partial r} \right) + \frac{\partial}{\partial z} \left( r \frac{\partial p}{\partial z} \right) = r \left( \left( \frac{\partial v}{\partial r} \right)^2 + 2 \frac{\partial u}{\partial r} \frac{\partial v}{\partial z} + \left( \frac{\partial u}{\partial z} \right)^2 \right) + \frac{v^2}{r} \]  \hspace{1cm} (9)

And the derivative of the pressure on the wall is expressed as follows by Pearson [25],

\[ \frac{\partial p}{\partial s} = -\frac{1}{Re} \frac{\partial \omega}{\partial n} \]  \hspace{1cm} (10)

where s and n are the tangential and normal directions respectively. The pressure on the wall can be time-dependently solved from Eqs. (9) and (10) by the Gauss-Seidel iteration method.
3.2 Calculation Steps

The computational scheme uses FTCS method (Forward-Time, Centered-Space). The finite differences for both the stream-function vorticity and the pressure drop are calculated by the Gauss-Seidel iterative method, respectively. The variables at \((m+1)\) iteration are calculated by using the values of the present iteration \((m+1)\) which have just been computed. The iteration number \(m\) takes an integer from 0 (initial value for the first iteration). The time step goes forward explicitly. The number of time step \(n\) takes an integer between 0 (initial time) and \(N\) (steady state) in Table 5.

Iteration independent values of the stream-function and vorticity are \(\psi^0\) and \(\omega^0\). Iteration dependent values are \(\tilde{\psi}^n_m\) and \(\tilde{\omega}^n_m\).

\(\psi^0\): Stream function at time\(=n*\Delta t\), iteration independent value.
\(\tilde{\psi}^n_m\): Stream function at time\(=n*\Delta t\), iteration dependent value.
\(\omega^0\): Vorticity at time\(=n*\Delta t\), iteration independent value.
\(\tilde{\omega}^n_m\): Vorticity at time\(=n*\Delta t\), iteration dependent value.

The calculation steps for the iterative method are summarized as the following Eqs. (11)–(15). \(f_1, f_2,\) and \(f_3\) are functions as to the stream-function and vorticity.

\[
\omega^0 = f_1(\psi^0) \quad (11)
\]
\[
\omega^{n+1} = \omega^n + \Delta t * f_2(\psi^n, \omega^n) \quad (12)
\]
\[
\tilde{\psi}^{n+1}_m = f_3(\tilde{\psi}^{n+1}_m, \tilde{\psi}^{n+1}_m, \omega^{n+1}) \quad (13)
\]
\[
\tilde{\omega}^{n+1}_m = f_1(\tilde{\psi}^{n+1}_m) \quad (14)
\]
\[
|\omega^{n+1} - \tilde{\omega}^{n+1}_m| < \varepsilon \quad (15)
\]

Setup:
1) \(\psi^0\) is given as the initial condition for all mesh points.
2) \(\omega^0\) at the boundaries is given as the initial condition. \(\omega^0\) inside the boundaries is calculated necessarily by the Poisson equation (11).

The time step \(n\) starts from 0.

Next step:
3) The iteration number \(m\) takes 0 for the first iteration. \(\omega^{n+1}\) at time\(=(n+1)*\Delta t\) are calculated necessarily by the vorticity transport equation (12) at time\(=n*\Delta t\).

Gauss-Seidel method:
4) \(\tilde{\psi}^{n+1}_m\) is calculated temporarily by the Poisson equation (13) using \(\tilde{\psi}^{n+1}_m, \tilde{\psi}^{n+1}_m,\) and \(\omega^{n+1}\). At the first iteration \((m=0)\), \(\tilde{\psi}^{n+1}_0\) takes \(\psi^0\).
5) Temporary vorticity, \(\tilde{\omega}^{n+1}_m\) is obtained from \(\tilde{\omega}^{n+1}_m\) necessarily by the Poisson equation (14) in order to check the convergence of the stream function at time\(=(n+1)*\Delta t\).
6) If \(\tilde{\omega}^{n+1}_m\) is not equal to \(\omega^{n+1}\) as Eq. (15), the calculation must be repeated for the convergence at time\(=(n+1)*\Delta t\) and then goes back to step 4. The iteration
number \( m \) is increased by 1.

7) If \( \dot{\omega}^{n+1}_m \) is equal to \( \omega^{n+1} \) as Eq. (15), the convergence is satisfied at 
\[ \text{time} = (n+1)\Delta t. \]
Furthermore, the steady state of the flow field is checked. When 
\( \omega^{n+1} \) and \( \omega^n \) are different, the flow field does not reach to the steady state. The time 
step \( n \) is increased by 1 and the calculation goes to next time step 3. On the contrary, if 
\( \omega^{n+1} \) and \( \omega^n \) become the same, the flow field reaches to the steady state and then the 
calculation stops.

3.3 Aspect Ratio and Time Increment

The velocity can be made dimensionless by dividing itself by a mean axial velocity. At 
the entrance length, the dimensionless axial velocity at the center line is just twice 
as large as the dimensionless average axial velocity for laminar flow, regardless of the 
Reynolds number. That is, the dimensionless difference of the axial velocity between 
the inlet and the entrance length is perfectly the same for the laminar flow, regardless 
of the Reynolds number. When the difference is divided by a constant axial grid 
number, a quotient becomes the same and remains constant. Therefore, when some 
dimensionless physical value is divided by a constant axial grid number, a difference is 
the same and remains meaningful for the laminar flow, regardless of the Reynolds-
number.

The entrance length is approximately proportional to the Reynolds number [26], 
that is, it becomes proportionally long as the Reynolds number increases. Thus, we 
need more and more axial grid points at high Reynolds numbers if the aspect ratio of 
d\( z \) to \( dr \) is the same for all Reynolds numbers. The axial space increment \( dz \) may be 
chosen to be proportional to the Reynolds number in order to keep the quotient 
constant for each Reynolds number. Consequently, when \( dr \) is constant, the aspect 
ratio of \( dz \) to \( dr \), which is denoted as DZR, becomes proportional to the Reynolds 
number.

It depends very strongly on the time increment whether time-dependent computa-
tional schemes may converge or diverge. When the flow field is computed 
time-dependently, at least several thousands time steps are required for each case.

The smaller the time increment is, the better convergence of computational 
numerical calculation is obtained. The time increment was derived from Eq. (16).

\[
\Delta t = \frac{Ct}{4 \left( \frac{1}{\Delta z^2} + \frac{1}{\Delta r^2} \right) + \frac{1}{\Delta z} + \frac{1}{\Delta r}}
\]

where \( Ct \) is a mere coefficient for the convergence. If the results of the numerical 
computation becomes divergent, \( Ct \) must be made smaller than the current value for 
the convergence of the computation. If the scheme continues to be in a convergent 
state until the steady state is reached, \( Ct \) can be made larger than the current value for 
the computational performance.

Table 6 gives the value of the time increment and the number of time step at which 
the steady state of the flow is obtained, with respect to the Reynolds number.

As the aspect ratio DZR increases with the Reynolds number, the derivatives with
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Table 6. Value of mesh system

<table>
<thead>
<tr>
<th>Re</th>
<th>IO</th>
<th>JO</th>
<th>DZR</th>
<th>dt</th>
<th>N*(dt)</th>
</tr>
</thead>
<tbody>
<tr>
<td>10</td>
<td>100</td>
<td>21</td>
<td>1</td>
<td>0.00278</td>
<td>2000</td>
</tr>
<tr>
<td>100</td>
<td>100</td>
<td>21</td>
<td>10</td>
<td>0.02621</td>
<td>8000</td>
</tr>
<tr>
<td>2000</td>
<td>150</td>
<td>21</td>
<td>100</td>
<td>0.04762</td>
<td>8000</td>
</tr>
<tr>
<td>10000</td>
<td>150</td>
<td>21</td>
<td>500</td>
<td>0.09901</td>
<td>20000</td>
</tr>
</tbody>
</table>

Fig. 6. Velocity distribution, Re=2000, DZR=10.

Fig. 7. Velocity distribution, Re=2000, DZR=50.

respect to $z$ decrease and become ineffective in the basic equations. However, in the DZR range of 1 and 500, the derivatives with respect to $z$ were meaningfully effective owing to double precision variables of IBM VS FORTRAN with 14 hexadecimal digits.

In this study, the values of the space increments in Table 6 are used in the computation. The aspect ratio depends on the Reynolds number and is proportionally chosen from 1 to 500. In order to confirm the accuracy of the computational results, the velocity distributions at Reynolds number of 2000 are calculated and compared with
one another in three cases of DZR = 10, 50, and 100. Figs. 6, 7 and 8 are the results of calculation in a steady state for DZR = 10, 50, and 100 respectively. As the axial grid number 10 remains constant, the computed region, which the computational mesh system covers, decreases with smaller DZR. Therefore, the maximum dimensionless axial length $z^*$ is 0.018 in the case of DZR = 10. In Figs. 6–8 ○ and ▲ show the results of the velocity distribution at $z^*$ = 0.01 and 0.05, respectively.

They agree very well with one another. Accordingly, we can use the large aspect ratio for the mesh system.

3.4 Velocity Distribution

The results are presented for the numerical solution of the complete, time-dependent equations of motion for four different Reynolds numbers of 10, 100, 2000, and 10000. If the number of time step reaches to the value N in Table 6, the flow field can be assumed to be in a steady state because variables at N time step is almost precisely the same as those at (N+1) time step.

Figs. 9–12 show the results for the axial velocity distribution in a steady state. At
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Fig. 10. Velocity distribution, Re=100, steady state.

Fig. 11. Velocity distribution, Re=2000, steady state.

Fig. 12. Velocity distribution, Re=10000, steady state.
Reynolds number of 10, in the central \(0<r<0.4\) and \(z^*<0.05\) region, the axial dimensionless velocities are fairly smaller than those of Reynolds numbers of 100, 2000, and 10000. In other words the flow field at Reynolds number of 10 develops more slowly than at Reynolds numbers of above 100. For instance, at the point of \(z^*=0.01\) and \(r=0\), the axial dimensionless velocity \(u/\nu\) is 1.3032 for \(Re=10\); 1.6289 for \(Re=100\), 1.6191 for \(Re=2000\) and 1.6191 for \(Re=10000\). At the point of \(z^*=0.05\) and \(r=0\), \(u/\nu\) is 1.9302 for \(Re=10\); 1.9735 for \(Re=100\), 1.9732 for \(Re=2000\) and 1.9733 for \(Re=10000\). The flow field in the entrance region develops completely fully near \(z^*=0.1\) and far downstream, that is, the velocity distribution after \(z^*=0.1\) is nearly the same for all Reynolds numbers. At the center line \(r=0\), \(u/\nu\) is 1.9977 (99.89% of fully developed value) for \(Re=10\); 1.9988 (99.94%) for \(Re=100\), 1.9929 (99.65%) for \(Re=2000\) and 1.9846 (99.23%) for \(Re=10000\).

So far various approximate analytical solutions have been devised in order to provide information relating to the flow development and the pressure drop in a circular pipe and a lot of experiments have been conducted by many researchers. Without experimental analyses there have been few analyses for high Reynolds numbers of more than 1000. Therefore, the numerical results at Reynolds number of 100 are compared with experimental data and with prior analyses.

Comparison with experimental data by Nikuradse [32] is well satisfied, although Nikuradse's entrance length is about 0.0625. Moreover, in Fig. 10 are the results of the numerical solution of the axial velocity component on the center line by Vrentas [41]. Vrentas calculated in the range of Reynolds numbers, based on the pipe diameter, of 0, 1, 50, 150, and 250. The comparison shows that the axial velocity distributions and the velocity development are nearly the same for Reynolds numbers of more than 50.

Table 2 shows some results of the numerical solution, according to which we can say in general that the dimensionless entrance length for 98% and 99% velocity development are 0.045 and 0.055, respectively, for Reynolds numbers of above 50.

3.5 Pressure Drop

Figs. 13–16 represent the results of the numerical simulation of the pressure drop on the wall in the entrance region. At Reynolds number of 10 there are strong vorticities on the wall just after the inlet, and large reverse pressure drop appears distinctly. The pressure drop is nearly the same for Reynolds numbers of above 100 and increases in the almost same degree, regardless of the Reynolds number. At Reynolds numbers of 10 and 100, the flow field develops almost fully after the 100*dt time step since the trends of the computational results of the pressure drop reach the steady state in Figs. 13–14. The steady state requires the computational time step of 8000*dt and 20000*dt for \(Re=2000\) and 10000, respectively.

Comparison of pressure drop: In the entrance region of the pipe, it is necessary to have a larger pressure drop per unit length than is required in the fully developed flow, since a part of this drop is utilized for accelerating the central core and consequently for increasing the kinetic energy of the flow. The excess pressure drop is the function of both the entrance distance from the inlet and the Reynolds number.
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Fig. 13. Pressure drop, Re=10, on wall.

Fig. 14. Pressure drop, Re=100, on wall.

Fig. 15. Pressure drop, Re=2000, on wall.
Fig. 14 shows the comparison of the experimental data by Kreith [17] and the results of the numerical simulation at Reynolds number of 100. They agree well with one another in a steady state, time=8000*dt. The numbers in Fig. 14 are Reynolds numbers of the experimental data measured by Kreith.

3.6 Axial Convective and Viscous Terms

The axial convective and viscous terms of two-dimensional Navier-Stokes equations are expressed in the cylindrical coordinates as follows,

Axial convective term:
\[
\nu \frac{\partial u}{\partial r} + u \frac{\partial u}{\partial z}
\]

Axial viscous term:
\[
\frac{1}{Re} \left( \frac{\partial^2 u}{\partial r^2} + \frac{1}{r} \frac{\partial u}{\partial r} + \frac{\partial^2 u}{\partial z^2} \right)
\]

The flow characteristics in the entrance region can be clearly seen through the Figs. 17–18, which are the results of the numerical simulation at Reynolds number of 10000 and \(z^*=0.0025, 0.01, 0.03, 0.05, \) and 0.1. The similar pattern of Figs. 17 and 18 is also seen for Reynolds numbers of above 10. The value of the axial convective term near the inlet is distinctly larger than that downstream, specially in central core \(0\leq r < 0.3\).

Subscripts 1–6 are given in Figs. 17 and 18.

1) The value of the axial convective term is almost zero near the wall after a short distance from the inlet \((z^* > 0.01)\). It becomes zero after \(z^*\) is larger than 0.03. The velocity distribution near the wall seems almost in a steady state after \(z^* > 0.03\), from Fig. 12. The axial convective term of zero means that the flow is in a steady state.

2 and 3) The value of the axial viscous term near the wall is about \(-0.004\) at \(z^*=0.01\) and its absolute value is almost the same as the value of the convective term in the central core.

4) The value of the axial viscous term in the central core stream is about zero only
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Fig. 17-a. Axial convective term, $Re=10000$.  

Fig. 17-b. Axial convective term, $Re=10000$.  

Fig. 18. Axial viscous term, $Re=10000$.  

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near the upper edge of a pipe, $z^*<0.01$. The condition for the boundary layer approximation, which is "the viscous term vanishes in the outer layer or the central core," is satisfied only in this short entrance region, $z^*<0.01$.

5) The value of the axial convective term decreases with the distance from the inlet and becomes equal to zero in every cross-section after $z^*=0.1$. Zero means that the flow field is in a fully developed state. Accordingly, the dimensionless entrance length is 0.1 for 100% velocity development.

6) The absolute value of the axial viscous term in the central core increases with the distance from the inlet and reaches to the steady state, which is the same as the value of the pressure drop.

Table 7 is the summary mentioned above as to the axial convective and viscous terms in the entrance region.

### 3.7 Comparison with N-S Equations in Steady State

For the steady two-dimensional Hagen-Poiseuille flow, the velocity components in the tangential and radial directions are zero; the pressure is constant in every cross-section. The Navier-Stokes equations simplify to only one equation in the standard form,

$$\frac{dp}{dz^{**}} = \frac{2}{Re} \left( \frac{d^2u}{dr^2} + \frac{1}{r} \frac{du}{dr} \right)$$

(17)

where $dz^{**}=dz/D$. Moreover, when the axial coordinate is made dimensionless by being divided by the Reynolds number, the equation is written as follows:

$$\frac{dp}{dz^*} = 2 \left( \frac{d^2u}{dr^2} + \frac{1}{r} \frac{du}{dr} \right)$$

(18)

where $dz^*=dz^{**}/Re=dz/(D*Re)$.

The velocity profile is the function of both the radius and the pressure drop,
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4. Definition of Problem of Transition

4.1 Stability Theorem of Lord Rayleigh

A problem identified is a problem half solved. Here, the problem is clarified and reasonably defined.

Lord Rayleigh [30] studied the inviscid stability of Poiseuille flow and concluded that the flow was stable to infinitesimal disturbances. He conjectured that the flow might be stable to infinitesimal disturbances but unstable to finite disturbances, or that the inviscid theory might be completely inapplicable to this problem. He derived several important, general theorems concerning the stability of laminar velocity profiles, and Tollmien verified these for more general conditions [32]. The validity of these theorems has been confirmed for both inviscid and viscous flows.

1) Theorem 1: The existence of a point of inflection constitutes a necessary condition for the occurrence of instability. Much later, Tollmien showed that it is also a sufficient condition for the amplitude of disturbances.

2) Theorem 2: The velocity of propagation of neutral disturbances in a boundary layer is smaller than the maximum velocity of the mean flow.

4.2 Schiller’s Observations on Transition

The problem was studied experimentally by Schiller [8] and the flow could be visualized by inserting a thin thread of dye into the fluid with a small pipette. For Reynolds numbers below 300, the stream line separating the deadwater region from the main flow appears. For Reynolds numbers greater than 1600–1700 the inlet flow takes on a new appearance and large, elongated vortices appear in the deadwater region. Schiller and Kurzweg suggested that the onset of turbulence occurred when

\[
\frac{D \Gamma}{2\nu} = 1170
\]

where \( \Gamma \) is the circulation per unit length near the wall.

4.3 Problems Proposed by Lin

According to Lin [22], most of the research work on the stability of laminar motions has the following final objectives:

1) The first aim is to determine whether a given flow (or a given class of flows) is ultimately unstable for sufficiently large Reynolds numbers. For this purpose, it is desirable to obtain some simple general criterion which will give a rapid classification of velocity profiles according to their stability.

2) The second purpose is to determine the minimum critical Reynolds number at which instability begins. It is often easier to find sufficient conditions for stability than
to find the condition for passage from stability to instability.

3) Finally, we want to understand the physical mechanism underlying the phenomena by giving theoretical interpretations and experimental confirmation of the results obtained from mathematical analysis.

4.4 Problems Observed by Leite

Leite's experiments were mostly carried out in the fully developed region. For that region, some important results on the stability of a circular pipe are stated as follows:

1) Axially symmetric Poiseuille flow was found to damp the small disturbances introduced, whether they be axially symmetric or not, up to a Reynolds number of 13000.

2) Experimental values of rate of decay were found to agree satisfactorily with those given by a recent theoretical analysis, even though assumptions of axial symmetry and longitudinal homogeneity of the disturbance are assumed in the latter.

3) To a first approximation, the propagation velocity of the disturbance does not depend upon radial position, hence upon local stream velocity, and is independent of distance downstream.

4) It has been found that small disturbances decay at all Reynolds numbers investigated and that large disturbances are unstable. Therefore, for fixed Reynolds numbers some disturbance of intermediate amplitude must be marginally stable.

4.5 Problems of Transition

Several results of prior experimental and analytical research are summarized graphically in Fig. 19 in such a way that immediate comparison can be easily made. Both dimensionless entrance and transition length are on a log scale. The entrance length is of Shiller, Langharr, Mohanty, Vrentas, and Kanda. Despite the fact that results of Sciller are the smallest among them, the order of each value is nearly the same and the largest dimensionless entrance length is calculated 0.1 by Kanda and Oshima. The experimental data is of Reynolds, and Oshima and Kanda. The relation between two experimental lines seems to depend on the shape and size of bellmouth. If a smaller bellmouth is used (case A), a graph of experimental data will be plotted between two lines. If a larger bellmouth is used (case B), a graph will be above the line with the bellmouth and shifted to the right. Different critical Reynolds numbers exist under different experimental apparatus. The transition length is strongly affected by the shape and size of bellmouth. Without bellmouth, a critical Reynolds number decreases to the minimum value, that is, the minimum critical Reynolds number which exists from about 2000 to 2300. Moreover, for the reproducibility of experimental data, it is very interesting that the results of Reynolds agree well with those of Oshima and Kanda where bellmouth was used: Let=0.00210 ar Re=13473. Reynolds experimented by using pipes with bellmouth, too. "The experiments were made on three tubes. The diameters of these were nearly 1 inch, 1/2 inch and 1/4 inch. They were all fitted with trumpet mouthpieces, so that the water might enter without disturbances."

The Reynolds' result is Let=0.00233 at Re=12900, which is very close to the
experimental line with the bellmouth. Moreover, it is found that all of the experimental transition lengths are shorter than any of the entrance lengths in the dimensionless form in Fig. 19. In addition, the results of stability analysis of Tatsumi and Huang are presented together in Fig. 19. The patterns seen in the down half of their curves of stability are similar to the graphs of experimental data. As the Reynolds number increases, the dimensionless transition length becomes shorter. The results of Tatsumi and Huang correspond to case A and B, respectively, and are also smaller than the dimensionless entrance length.

As a result of the above discussion the author concludes the problem of the transition as follows:

1) The transition occurs in the entrance region. The transition length becomes shorter as the Reynolds number increases under the same experimental apparatus.

2) A critical Reynolds number depends on an experimental apparatus, especially on the shape of bellmouth, provided not external nor manual disturbances are given.
to the upstream and inlet conditions.

3) The minimum critical Reynolds number exists under the condition of infinitesimal disturbances (i.e. a case without bellmouth).

5. **Simulation of Transition**

5.1 **Simulation Cases**

The Navier-Stokes equations can be regarded as the basis, rather than a set of equations derived for stability analyses. There is no fundamental reason why one should not be able to simulate a turbulent flow in any desired detail. The Navier-Stokes equations are believed to be a sufficient description, and discretization errors can in principle be made as small as one wishes. The convergence of iterative methods is affected by several parameters, such as the Reynolds number, mesh spacing, time increment, and especially the iteration convergence criteria. One of the most interesting problems concerns the selection of appropriate conditions for the inlet boundary. There can be assumed many types of disturbances which exercise a marked effect upon the transition from laminar to turbulent flow. We do not know precisely the magnitude of infinitesimal disturbances of real flows. For example, Leite measured the mean amplitude of axial component of small residual disturbances and found that it had a maximum value of approximately 10**(−4). These disturbances were believed to be largely caused by radial sound, because their amplitudes varied little across the pipe. Here, simulations for finite disturbances are discussed. Accordingly, we can roughly estimate the characteristics of the flow field from the result of vorticity and velocity profile, although it is difficult to simplify and summarize the enormous volume of results produced.

The objective of numerical simulation is to directly simulate the transitional flow for the case of the inlet without bellmouth. Some disturbances given just at the inlet are investigated whether they decay or grow. Thus, the transition lengths at Reynolds numbers of 2700 and 10000 are the main goal to be obtained in the case without bellmouth. The experimental data measured by the color-dye method are:

Let=0.00234 at Re=2702, Let=0.00006 through 0.0001 at Re=10000.

The transition point for numerical analysis is the point where the limiting stream line changes to back flow, that is, the first separation occurs and back flow appears on the wall. The point at which the flow separates from the wall depends on the Reynolds number. The higher the Reynolds number, the sooner the flow separates.

The simulation of disturbed flow fields is performed for six cases as summarized in Table 8. Each point in the pipe is expressed as Pc(z*, r) in the cylindrical coordinates or as Pm(i, j) in the mesh system. The relations between Pc and Pm are: z*=(i−1)*dz, r=(j−1)*dr, where dr=0.5/20=0.025.

Concerning the magnitude and location of disturbance, for example, in case 1 and 2, 1.2 times singular stream function is given at two points, Pm(1, 11) and Pm(2, 11). This type of disturbance is denoted as 2A*1.2, where “2A” is the number and position of disturbed mesh points, Pm(1, 22)=Pc(0, 0.25) and Pm(2, 11)=Pc(0, 0.25), and “1.2” is the value of amplitude constant, A, for the stream function. The stream
function at \( Pm(1, 11) \) and \( Pm(2, 11) \) is multiplied by amplitude constant of 1.2. However, the stream function at the outlet wall is constant, regardless of time step, since the total flux across a section must be constant throughout the pipe. The amplitude constant ranges from 1.005 to 1.2. Disturbances are given in the central core stream or near the wall of the inlet part of the pipe. The disturbance “A” is for the central core. “B, C, and D” are for points near the wall. \( Pm(1, 19) \), \( Pm(1, 18) \), and \( Pm(1, 17) \) are points of \( 2dr, 3dr \), and \( 4dr \) apart from the wall: \( Pm(1, 19)=Pc(0, 0.45), Pm(1, 18)=Pc(0, 0.425) \), and \( Pm(1, 17)=Pc(0, 0.4) \). “1B” consists of \( Pm(1, 18) \), “2C” of \( Pm(1, 19) \) and \( Pm(1, 18) \), and “3D” of \( Pm(1, 19) \), \( Pm(1, 18) \) and \( Pm(1, 17) \).

In regard to the aspect ratio, large aspect ratio, which is used in section 3, is given for case 1, 2 and 3. On the other hand, small aspect ratio is used in case 4, 5, and 6 in order to catch mesh-size disturbances, where the mesh spacing is each constant and it is 1 for cases of \( Re=10000 \) and 2 for \( Re=2700 \). Time increment for cases of 1, 2, and 3 is given in Table 6. It is 0.01 for cases of 4, 5, and 6. In each case it is made dimensionless by dividing by \((R/\omega_0)\).

The initial and boundary disturbances in cases of 4, 5, and 6 are given as follows:

1) First, the stream function is given as all the fluid particle start moving downstream with uniform velocity as Eq. (20).

2) Values of the stream-function are multiplied by an amplitude constant for particular points as Eq. (21).

3) After starting (time=\( dt \)), the same disturbances at the inlet remain constant, but the initial disturbances of other points are freed. Stream-function for these points are calculated according to a scheme as Eq. (22).

\[
\psi(i, j) = \frac{1}{2} \left( \frac{j-1}{JO-1} \right)^2 \quad \text{for } i=1\sim IO \text{ and } j=1\sim JO
\]  
(20)

\[
\psi(i, j) = \psi(i, j) \cdot Am \quad \text{for } i=1\sim IO \text{ and } J=JO-2, JO-3, \text{ and } JO-4
\]  
(21)

\[
\psi(i, j) = \psi(1, j) \cdot Am \quad \text{for } j=JO-2, JO-3, \text{ and } JO-4
\]  
(22)

where \( Am \) is the amplitude constant.
For smaller disturbances, the amplitude constant and number of stressed grid points will be decreased accordingly.

5.2 Effect of Disturbances at $Re=2000$ and $10000$ (2A*1.2)

In this section, 1.2 times singular stream-function were given at two inlet and central points, $z=0$, $dz$, and $r=0.25$. It was simulated whether the singularity of the velocity distribution would be amplified or would be damped as the time step develops. Figs. 20 (a)-(c) are the results of the numerical simulation at $Re=2000$ and $z=dz$ ($z^* = 50/(40*2000) = 0.000625$), $z=2dz$ ($z^* = 0.00125$), and $z=4dz$ ($z^* = 0.0025$). Figs. 21 (a)-(c) are the results at $Re=10000$ and $z=dz$ ($z^* = 500/(40*10000) = 0.00125$), $z=2dz$ ($z^* = 0.0025$), and $z=4dz$ ($z^* = 0.005$). The singularity of the velocity distribution seems to be damped smoothly at $Re=2000$ after $time=500dt$. At $Re=10000$, the singularity seems to be damped after $z^* = 4dz$. However, the strong deformation of the velocity distribution still remains at $z = dz$ after $time=4500dt$.

An inflection point near the wall does not exist in both cases.

Fig. 20-a. Velocity distribution, $Re=2000$, Dist=2A*1.2, time=10dt.

Fig. 20-b. time=50dt.

Fig. 20-c. time=500dt.
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Fig. 21-a. Velocity distribution, Re=10000, Dist=2A*1.2, time=10dt.

Fig. 21-b. time=50dt.

Fig. 21-c. time=4500dt.
5.3 Effect of Disturbances at Re=10000 (1B*1.2)

In order to simulate the inflection point near the wall, which is the necessary condition for the occurrence of transition, disturbances are given for a point of r=0.425 and z*=0 near the wall when using the large aspect ratio of 500. Fig. 22 is the result in a steady state at time=20000dt.

Compared with Fig. 21 (c), the velocity distribution is minus at the inlet and very close to the wall (z*=0 and r=0.475) as shown in a circle. After the inlet, however, reverse velocity distribution cannot seen close to the wall. We should simulate the disturbed flow field using a mesh system with smaller aspect ratio.

5.4 Effect of Disturbances at Re=2700 (2C*1.005)

The flow field in the entrance region is simulated at a Reynolds number of 2700. A small disturbance of 0.5% of a stream function is added to two points: r=0.045, 0.425 and z*=0.

Iteration convergence did not attain an acceptable solution to the discretized difference equations after time=2473dt. The differences of the vorticity between iteration m and m+1 do not fall within tolerance despite many iterations at time=2474dt. The results at time=2200dt and 2469dt are mainly discussed.

5.4.1 Effect of Disturbances upon Velocity Development

The velocity developments along the pipe are shown in Figs. 23 (a)-(d) for r=0.475, 0.45, 0.35, and 0. The wall and center line are at r=0.5 and 0, respectively. The velocity distributions in the central core (r=0.35 and 0) oscillate considerably, although those near the wall (r=0.475 and 0.45) appear to still remain laminar. Fig. 23 (c) shows that the fluctuation of the velocity at r=0.35 begins at about z*=0.0005 (27dz). That at the center line is a little delayed in the beginning (z*=0.0007 (38dz)) from Fig. 23 (d). The amount of amplification decays rapidly after z*=0.003 (162dz). This does not mean that the flow changes from turbulent to laminar after that point. Rather, this is because the inlet disturbances are carried approximately 162dz downstream for 2200 time-steps of computation, and not transferred after 163dz.

Fig. 22. Velocity distribution, Re=10000, time=20000dt, Dist=1B*1.2.
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Fig. 23-a. Velocity development, Re=2700, time=2200dt, r=0.475, Dist=2C*1.005.

Fig. 23-b. r=0.45.

Fig. 23-c. r=0.35.
downstream point. Unfortunately, the computation cannot develop further under the given iteration convergence criteria.

Supposing the velocity of propagation of neutral disturbances is defined as,

\[
\text{Velocity of propagation of neutral disturbances} = \frac{\text{axial disturbed length of flow field}}{\text{time}}
\]

its value is calculated from Figs. 23 (c) and (d).

\[
= \frac{162dz}{2200dt} = \frac{162 \times (1/20) \times 2}{2200 \times 0.01} = 0.736 < 1
\]

where space and time increments, and aspect ratio are 1/20, 0.01, and 2, respectively. In general, the radial space increment is 0.5/20=0.0025. However, here, the radial space increment (1/20) is made dimensionless by dividing by the radius of the pipe (R) since the time increment is made dimensionless by dividing by (R/uo). A maximum
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Fig. 24-b. \( r=0.35 \).

Fig. 25. Velocity development, \( Re=2700, \) time=2200\( dt, \) \( r=0.475 \) and 0.35, Dist=no.

velocity is about 1 in a dimensionless form. This result seems to be reasonable for Rayleigh's Theorem 2 (see 4.1).

Figs. 24 (a)-(b) also display the velocity development at time=2469\( dt \). The separation point on the wall can be clearly seen at \( z^*=0.00185 (100dz) \) in Fig. 24 (a) and sharp back flow is observed at the same longitudinal distance in Fig. 24 (b). These values are only a little smaller than the experimental longitudinal value of 0.00234 from Table 3. The order of both values, however, are nearly the same. The error is \(-20.9\%: \frac{0.00185-0.00234}{0.00234}=-0.209\).

Compared to a case without any disturbance, the velocity developments at \( r=0.475 \) and 0.35 are displayed in Fig. 25. These lines develop smoothly without oscillating. The average value of Fig. 23 (c) is, of course, equal to that in Fig. 25 (\( r=0.35 \)).

5.4.2 Effect of Disturbances upon Velocity Distribution

Figs. 26 (a)-(g) display how the velocity distribution is influenced by the inlet disturbances over cross sections perpendicular to a longitudinal direction: \( z^*=0, \) 0.000259 (14\( dz \)), 0.000759 (41\( dz \)), 0.0015 (81\( dz \)), 0.00185 (100\( dz \)), 0.002 (108\( dz \)), and
Fig. 26-a. Velocity distribution, $Re=2700$, $time=2200dt$, $Dist=2C*1.005$, $z^*=0.0$ (0dz).

Fig. 26-b. $z^*=0.000259$ (14dz).

Fig. 26-c. $z^*=0.000759$ (41dz).
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Fig. 26-d. \( z^* = 0.0015 \) (81dz).

Fig. 26-e. \( z^* = 0.00185 \) (100dz).

Fig. 26-f. \( z^* = 0.002 \) (108dz).
0.003 (163dz), respectively. The velocity fluctuations given as initial and boundary conditions are shown in Fig. 26 (a), and are not too much larger than expected. Larger fluctuation is given in 5.5.2. The velocity distribution starts waving up and down after \( z^* = 0.000759 \) in Fig 26 (c), although it develops in a laminar state until about 0.0007 from Fig. 23 (c). Further downstream in Fig. 26 (g), the velocity distribution regains a laminar flow. The highest value of a periodically varying velocity is seen at about \( z^* = 0.00185 \) in Fig. 26 (e). The trend of a varying, axial velocity is well understood, even though its value is considerably higher than that of a real flow. The inflection point, which is a necessary condition for the occurrence of instability (Rayleigh's Theorem 1), is not still perceived clearly near the wall. On the contrary, it exists in the central core, as shown in Fig. 26 (e).

5.4.3 Effect of Disturbances upon Vorticity

Vorticity is made dimensionless by dividing by \( (u_0/R) \).

Figs. 27 (a)-(g) display the transfer of vorticity in the entrance region of the circular pipe with time, for time = 10dt, 500dt, 1500dt, 1700dt, 1800dt, 2200dt, and 2300dt,
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Fig. 27-b. time=500dt.

Fig. 27-c. time=1500dt.

Fig. 27-d. time=1700dt.
Fig. 27-c. time = 1800dt.

Fig. 27-f. time = 2200dt.

Fig. 27-g. time = 2300dt.
respectively. In general, the vorticities near the wall are stronger than those in the central core before time=2200dt and their value lies between −9 and 37. The reverse vorticity appears only after time=2200dt. The inlet disturbances produce stronger vorticities in the central core than near the wall in Fig. 27 (g). The residual errors for the vorticity become larger with time until convergent solutions cannot be obtained. They are much amplified around the small region after time=2300dt, where the dimensionless axial length \( z^* \) is about 0.0018 in Fig. 27 (g). The disturbances used have a strong influence on the vorticity of this region. Figs. 27 (c)-(e) show explicitly how the vorticities are carried to the central part. A vorticity line with equal strength separates from the wall to the central core, then it turns round and round, and finally it breaks into many pieces of vorticity.

5.5 Effect of Disturbances at \( Re=10000 \) (3D+1.1)

The flow field is simulated at a high Reynolds number of 10000. The 10% increased values of a stream function are supposed at three points: \( r=0.045, 0.425, 0.4 \) and \( z^*=0.00015 \). These disturbances are considerably larger than in 5.4. Iteration convergence did not attain an acceptable solution after time=370dt and results are saved on a disk-file every 10dt. Accordingly, the velocity distributions at time=350dt are presented. We can roughly extrapolate, from Fig. 19, that a dimensionless transition length is about 0.00006 to 0.0001 at \( Re=10000 \). Its value corresponds to 1.8 to 3 cm in the case of the pipe of 3 cm in diameter and without bellmouth: 0.00006×10000×3=1.8. A flow changes from laminar to turbulent very near the inlet of the pipe at Reynolds numbers of above 10000.

5.5.1 Effect of Disturbances upon Velocity Development

The development of the axial velocity fields are illustrated in Figs. 28 (a)-(d) for several radii: \( r=0.475, 0.45, 0.35, \) and 0.25, respectively. In general, the amount of amplification decays rapidly after \( z^*=0.00015 \) (60dz).

\[
\text{Velocity of propagation of neutral disturbances} = \frac{60\text{dz}}{350\text{dz}} = \frac{60×(1/20)}{350×0.01} = 0.857 < 1
\]

This result seems to be reasonable for Rayleigh's Theorem 1. Fig. 28 (a) shows the velocity development at one dr away from the wall \( (r=0.475) \). There are three separation points on the wall, \( z^*=0.0000225 \) (9dz), 0.0000325 (13dz), and 0.000045 (18dz). The value for \( z^*=0.0000325 \) is the largest among them. Similarly, a big backflow also can be seen at \( z^*=0.000025 \) (10dz) in Fig. 28 (b). These values are about one third of the extrapolated value of the experimental data in Fig. 19, (about 0.00006 to 0.00001). The order of both values, however, is nearly the same. The error is −62.5%: (0.0000225−0.00006)/0.00006 = −62.5. These disturbances affect considerably the velocity at \( r=0.35 \) in Fig. 28 (c), but do not yet influence the velocity at \( r=0.25 \) in the more central core in Fig. 28 (d). The velocity fluctuation is the largest at \( r=0.35 \) among them.
Fig. 28-a. Velocity development, Re=10000, time=350dt, r=0.475, Dist=3D*1.1.

Fig. 28-b. r=0.45.

Fig. 28-c. r=0.35.
5.5.2 Effect of Disturbances upon Velocity Distribution

Figs. 29 (a)-(g) display how the velocity distribution grows under the inlet disturbances over cross sections perpendicular to a longitudinal direction: \(z^* = 0, 5dz, 9dz, 10dz, 11dz, 50dz, \) and \(98dz,\) respectively. The velocity fluctuation, given as initial and boundary conditions, is shown in Fig. 29 (a), and are similar to that at \(z^* = 98dz\) in Fig. 29 (g) since the disturbances do not grow after \(z^* = 60dz.\) The amount of fluctuation still remains larger at \(z^* = 50dz\) in Fig. 29 (f) than at the inlet in Fig. 29 (a). On the other hand, the velocity distributions without any disturbances are displayed for \(z^* = 0\) and \(0.000025 \text{(10dz)}\) in Fig. 30, and both lines show that a flow is in a laminar state. Backflows are also shown at \(z^* = 0.0000225 \text{(9dz)}\) and \(0.000025 \text{(10dz)}\) in Fig. 29 (c) and (d), respectively, even though the amount of fluctuation is considerable in Fig. 29 (d). The velocity profile near the wall shows the existence of an inflection point at \(z^* = 5dz \text{(0.0000125)}\) in Fig. 29 (b), which is a necessary condition for the occurrence of instability (Rayleigh’s Theorem 1).

The initial fluctuation in Fig. 29 (a) appears to be considerably larger than that in a
Fig. 29-b. $\mu = 0.0000125 (5dz)$.

Fig. 29-c. $\mu = 0.0000225 (9dz)$.

Fig. 29-d. $\mu = 0.000025 (10dz)$. 
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Fig. 29-c. \( z^* = 0.0000275 \) (11dz).

Fig. 29-f. \( z^* = 0.000125 \) (50dz).

Fig. 29-g. \( z^* = 0.000245 \) (98dz).
real flow, since it is about 1.4 times as large as in Fig. 30, which is the result in the case of no disturbance.

5.5.3 **Effect of Disturbances upon Vorticity**

Figs. 31 (a)-(f) display the transfer of vorticity for time=10dt, 100dt, 200dt, 300dt, 350dt, and 370dt, respectively. The value of vorticities is between $-11$ and $36$ before time=200dt, then its absolute value increases and reaches between $-976$ and $1454$ at time=370dt in Fig. 31 (f). However, the value at time=350dt ($-184$ to $304$) does not greatly exceed the average value before time=200dt. The residual errors for the vorticity are amplified around the small region at $z^*=0.00003$ and time=370dt. The disturbances used have a strong influence on the vorticity at about $z^*=0.00003$. Fig. 32 displays the vorticity in the case of no disturbance at 350dt, which shows a steady and laminar development of the flow field.
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Fig. 31-b. time=100dt.

Fig. 31-c. time=200dt.

Fig. 31-d. time=300dt.
Fig. 31.e. time=350dt.

Fig. 31.f. time=370dt.

Fig. 32. Vorticity, Re=10000, time=350dt, Dist=no.
5.6 Effect of Disturbances at Re=10000 (2C*1.01)

When the convergence criteria are better selected, doubtlessly the calculation step of the simulation will become longer. A smaller disturbance also makes a computation step longer. Initial and boundary disturbances can be easily controlled by an amplitude constant. Here, as a disturbance of smaller than 10% in 5.5, the 1% increased stream function is given at the same two points as in 5.4: r=0.045, 0.0425 and z* =0.

Iteration convergence did not continue after time=500dt, but it is longer than 370dt in 5.5. Consequently, the time step of calculation advances further with a smaller disturbance.

5.6.1 Effect of Disturbances upon Velocity Development

Figs. 33 (a)-(d) display the velocity development along the pipe for r=0.475, 0.45, 0.35, and 0.25 at time=500dt. The velocity distribution oscillates violently near the wall at r=0.475 to 0.35. On the contrary, it does not swing up and down with a regular

![Graph](image1)

Fig. 33-a. Velocity development, Re=10000, time=500dt, r=0.475, Dist=2C*1.01.

![Graph](image2)

Fig. 33-b. r=0.45.
motion near the central core at $r=0.25$ in Fig. 33 (d). This shows that iteration convergence does not arrive at a solution of finite-difference equations before the disturbances are transported to the central core. Fig. 33 (a) expresses clearly the separation points at $z^*=0.000085$ (34dz), 0.0001 (40dz), and 0.0001125 to 0.000115 (45–46dz). The dimensionless transition length agrees fairly well with the experimental value of 0.00006 to 0.0001 and is about four times as large as in 5.5.1 ($z^*=0.000025$). Figs. 33 (b)-(c) also show big back-flows near the wall. The amount of fluctuation decays after $z^*=0.0002$ (80dz).

\[
\text{Velocity of propagation of neutral disturbances} = \frac{80dz}{500dt} = \frac{80 \times (1/20)}{500 \times 0.01} = 0.8 < 1
\]

This result satisfies Rayleigh's Theorem 2.
5.6.2 Effect of Disturbances upon Velocity Distribution

Figs. 34 (a)-(d) display the velocity distribution over cross sections for $z^* = 0$, 9dz, 10dz, and 11dz, respectively. The velocity fluctuation, given as initial and boundary conditions, is shown in Fig. 34 (a). Its value is nearly equal to that in Fig. 26 (a), although the amplitude constant is 1.005 for the latter. Figs. 34 (b)-(d) are compared with Figs. 29 (c)-(d), respectively. The separation points already exist at $z^* = 0.000025$ in Fig. 29 (c). However, in the case of smaller disturbances, those appear more downstream first at $z^* = 0.000085$. We can see an inflection point near the wall in Fig. 34 (d).

5.6.3 Effect of Disturbances upon Vorticity

Figs. 35 (a)-(f) display the vorticity distribution in response to smaller amplitude disturbances. Vorticities are carried more downstream than in Fig. 31 (f). The vorticity starts flowing downstream at about $z^* = 0.00002$ in Fig. 35 (f) and its value increases with this downstream flow; this result meets more natural conditions than
Fig. 34-c. $z^*=0.000025$ (10dz).

Fig. 34-d. $z^*=0.0000275$ (11dz).

Fig. 35-a. Vorticity, $Re=10000$, time=$10dt$, Dist=$2C*1.01$. 
that of Fig. 31 (f). The highest value lies at about $z^*=0.0001$.

Fig. 35-b. time=100dt.

Fig. 35-c. time=200dt.

Fig. 35-d. time=300dt.
6. Conclusions

An entrance model was presented in order to numerically simulate the flow characteristics such as velocity distribution, pressure drop, convective and viscous terms for four different Reynolds numbers of 10, 100, 2000, and 10000. The following results are obtained.

1) The aspect ratio of axial to radial space increments DZR (dz/dr) can be provided proportional to the Reynolds number below 10000 when variables have double precision in FORTRAN. For instance, the ratio is 1 at $Re=10$; 10 at $Re=100$, 100 at $Re=2000$ and 500 at $Re=10000$. The results of the numerical computation in three cases of DZR=10, 50, and 100 are precisely the same one another for Reynolds number of 2000.

2) Smooth, two-dimensional, time-dependent, numerical solution of the Navier-Stokes equations exists regardless of the Reynolds number.

3) The dimensionless velocity distributions and pressure drops in the entrance
region are similar for Reynolds numbers of more than 100.

4) The dimensionless entrance length was calculated to be 0.1 for Reynolds numbers above 10. The dimensionless entrance lengths for 98% and 99% velocity development are 0.045 and 0.055, respectively, for Reynolds number of above 50.

5) The entrance region of $z^*<0.01$ is satisfied for the boundary approximation.

Moreover, the flow field of the circular pipe was studied with particular emphasis on the turbulent transition length from both the experiment and the direct numerical simulation. The results of this study show the numerical, finite-difference method can simulate the development of the transfer to turbulent flow.

6) The turbulence transition occurs only within the entrance region.

7) The transition length decreases as the Reynolds number increases under the same inlet and experimental conditions.

8) A good bellmouth lessens disturbances at the inlet and therefore the critical Reynolds number becomes large.

9) The transition was numerically simulated on the assumptions: a) the aspect ratios of the rectangular mesh system used are 2 for $Re=2700$ and 1 for $Re=10000$ and b) the disturbances are given at the point very near the wall of the inlet. However, in the cases of large DZR, the transition could not be simulated since large space increments exceed the size of disturbances.

10) The results of the simulation satisfy two theorems of Lord Rayleigh, as to the dependence of the flow stability on laminar velocity profiles.

11) The transition length depends on the magnitude of the disturbance assumed, that is, the inlet velocity distribution.

Consequently, in the case of large disturbances, the transition length obtained numerically is a little shorter than the experimental data, but considered fairly reasonable since the order of the values are the same. In the case of finite but small disturbances, the transition lengths obtained numerically are the same as the experimental ones and the vorticities flow downstream.

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