Theory of Relativistic String and Super-Wave Equation. I

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In this and succeeding papers we reconstruct the theory of relativistic string, taken as a model of hadrons, in a general and unifying framework. In this paper the classical foundation of the "realistic" string theory is clarified. The theory is derived from the action integral which is invariant under "partial general transformation" on the parameters $(\sigma, \tau)$ for the world sheet traced by a motion of the string, and the fundamental equations are originally covariant under this group. Then by choosing a suitable gauge we obtain Lorentz-covariant formalism associated with constraints. We analyze its characteristics and obtain solutions. Taking another gauge we get the equivalent "Lorentz non-covariant formalism" free from constraint. This directly exhibits the physical meaning of the theory and also leads to its standard Hamiltonian formalism. Several conservation laws characteristic to our string theory are given.

§ 1. Introduction

For various reasons theory of relativistic extended model for hadrons has been pursued for some years. This has been done in particular along the line of Yukawa's bilocal model and its multilocal generalization. Generally speaking a relativistic extended system cannot be a rigid body because of relativity, but it must be a certain elastic system with internal cohesion in order that it remains a finite continuum in course of its movement. Thus its size must depend on the state of motion and may vary during the motion. A simple and primitive example of such a relativistic extended model should be a one-dimensional elastic continuum ("string"). Indeed this is in a sense the most natural extension of the traditional point model which has hitherto underlied the usual theory of elementary particles. At the same time the relativistic theory of such a continuum is an interesting problem in its own right.

Some time ago it was indicated that a string-like model has the capability of deriving the Veneziano amplitude and its multiparticle generalization, which had just been proposed from $S$-matrix point of view for satisfying the concept of "duality" in hadronic reactions. Then we directly constructed the complete relativistic quantum mechanics of a finite one-dimensional elastic continuum, taken as a model of hadrons, in representing it by the set of wave equation and an infinite number of subsidiary conditions, or equivalently by a super-wave equation (called "detailed wave equation"). It was still relevant to reconstruct this theory by establishing first the classical theory of a relativistic string explicitly and then quantizing it, because this makes the correspondence-theoretical
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foundation and the realistic interpretation of the theory clearer, and indeed such reconstruction has been pursued by several authors.5-14 In the present and succeeding papers we reconstruct the theory in a more general and unifying framework and give deeper analysis in the theoretical foundation as well as concrete treatment of the string's movement. This paper is the first part of the work where we establish the relativistic string theory from the "realistic" viewpoint. In this viewpoint the theory is derived from a Lagrangian containing two structure constants. The equation of motion is originally covariant under "partial general transformation", and each elementary constituent (say "parton") of the string has three degrees of freedom. Then by choosing a suitable gauge we obtain "Lorentz-covariant formalism", which is convenient for quantization. Taking another gauge we get the equivalent "Lorentz non-covariant formalism", which is also useful because it directly exhibits the physical interpretation of theory.

We may construct a relativistic string theory also from "geometric" viewpoint, leading to a more particular model. This is given in a succeeding paper 15 (quoted as II hereafter), where the relationship between both string theories is also clarified.

§ 2. Relativistic string theory in the realistic viewpoint

A motion of a finite open string sweeps a two-dimensional timelike world strip bounded in the spacelike direction but unbounded in the timelike direction. We represent it as

\[ x_\mu = x_\mu (\sigma, \tau) \]  \hspace{1cm} (2.1)

with the aid of two independent parameters \( \sigma \) and \( \tau \), where \( \sigma \) runs over a finite domain \([\sigma_0, \sigma_1]\), while \( \tau \) over \((-\infty, \infty)\). Here \( \sigma_0 \) and \( \sigma_1 \) label the string's ends such that \( x_\mu (\sigma_0, \tau) \) and \( x_\mu (\sigma_1, \tau) \) represent the world lines of the ends. If \( (\sigma, \tau) \) are merely arbitrary Gauss coordinates specifying a point ("event") on the strip, there is the arbitrariness of general transformation

\[ \sigma \rightarrow \sigma'(\sigma, \tau), \quad \tau \rightarrow \tau'(\sigma, \tau); \quad \frac{\partial \sigma'(\sigma, \tau)}{\partial \tau} \bigg|_{\sigma_0, \tau_1} = 0 , \]  \hspace{1cm} (2.2a, b)

because the new \( (\sigma', \tau') \) give an equivalent representation of the same strip such that \( x_\mu = x_\mu'(\sigma', \tau') = x_\mu(\sigma, \tau) \). The condition (2.2b) is necessary because we always demand that the boundaries map the boundaries. The geometrical quantities intrinsic to the world strip are the line element on it \( (dx_\mu)^2 = G_{\alpha \beta} dx^\alpha dx^\beta \) and the surface element \( \sqrt{-D_\gamma d\sigma d\tau} \), where

\[ D_\gamma = G^{11}G^{20} - (G^{12})^2; \quad G^{\alpha \beta} = g_{\mu \nu} \frac{\partial x^\mu}{\partial \zeta_\alpha} \frac{\partial x^\nu}{\partial \zeta_\beta} \quad (\zeta_1 = \sigma, \zeta_2 = \tau), \quad \alpha, \beta = 1, 0 \]

The Minkowski metric \( g_{\mu \nu} \) is \( g_{\mu \nu} = \delta_{\mu \nu} (1, 1, 1, -1) \), whereas \( G^{00} = (\partial x_\mu / \partial \tau)^2 \).

\( ^* \) We employ the notation convention such as \( (dx_\mu)^2 = dx_\mu dx^\mu = \sum_i (dx_i)^2 - (dx_0)^2 \).
G^{11} = (\partial x_{\mu}/\partial \sigma)^2 \text{ and } G^{10} = (\partial x_{\mu}/\partial \sigma)(\partial x^{\tau}/\partial \tau) \text{ are the first fundamental quantities of the surface, forming the fundamental tensor of a 2-dimensional Riemannian space, and have to satisfy}

\[ D_5 \leq 0, \] 

(2.3)

because the strip must be timelike. At any point on the strip there exist two light-like directions, lying on the strip. They are given by

\[ d\tau/d\sigma = (-G^{10} \pm \sqrt{-D_5})/G^{00}. \]

Up to this point the statements are very general, but to proceed further we need to distinguish between the "geometric viewpoint" and the "realistic viewpoint". In the former viewpoint, only the world strip together with its two boundary world lines (representing the motions of the ends) has physical sense such that one world strip is regarded as representing one and the same motion of the string, whereas in the latter viewpoint one and the same world strip corresponds to many different motions of the string due to the situation that the same strip can be differently woven by world lines. The relativistic theory in the geometric viewpoint yields a unique string model where longitudinal motion along the string has no physical sense. (See II.) This model is favoured recently, but obviously it is not true to say that this unique model should be the only string model allowed by relativity. In this paper we take the "realistic viewpoint", which turns out to allow a more general relativistic model.

In the realistic viewpoint $\sigma$ in (2.1) denotes the parameter to label each elementary constituent of the string, and hence it must always be a Lorentz scalar. The strip is interpreted as a bunch of world lines, respectively corresponding to $\sigma = \text{const}$. (This interpretation assumes the distinguishability among constituents at least in classical theory.) To preserve this meaning of $\sigma$ the arbitrariness of the parameters $(\sigma, \tau)$ is limited within the transformation

\[ \sigma \rightarrow \sigma'(\sigma), \quad \tau \rightarrow \tau'(\sigma, \tau), \] 

(2.4)

which is a subgroup of (2.2). Even more specifically we may consider the string as the $N \rightarrow \infty$ limit of the linear multilocal model which consists of $N$ spacetime points $x^{(\alpha)}_{\mu}(\alpha = 1, \cdots, N)$ subject to "relativistic Hooke potential" between neighbours.\(^5\),\(^6\) Then $\sigma$ is the relabelling of $\alpha$ by $\sigma = (\sigma_1 - \sigma_0)\alpha/N$, and we can interpret that $d\sigma/(\sigma_1 - \sigma_0)$ is the relative particle number (or say "parton number") contained in $d\sigma$. In so far as we keep this physical meaning of $\sigma, \sigma$ is essentially fixed and we are not allowed to make an arbitrary transformation on $\sigma$. Therefore we agree to take $\sigma$ as dimensionless and $0 \leq \sigma \leq \pi$. On the other hand $\tau$ is an arbitrary parameter to specify a point along each world line. Thus

any observable quantities and physical relations must be invariant under the "partial general transformation"

\[ \tau \rightarrow \tau'(\sigma, \tau), \quad (\sigma' = \sigma) \] 

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Note that \( \tau \) need not be a Lorentz scalar and that (2.5) contains a transformation which alters the dimension of \( \tau \).

It is important to note that the arbitrary parameter \( \tau \) is also a redundant parameter so that it must be possible to eliminate \( \tau \) without leading to any ambiguity in the physical interpretation.\(^1\) To see this we rewrite (2.1) as \( x_i = x_i(\sigma, \tau) \) and \( x_0 = x_0(\sigma, \tau) \), and put \( x_0 = x_0(\sigma, \tau) = t \). Then by solving this and inserting the result into \( x_i = x_i(\sigma, \tau) \) we get the ordinary (non-covariant) representation for a string's motion

\[
x_i = x_i(\sigma, \tau(\sigma, t)) = x_i(\sigma, t),
\]

which is unique in so far as the arbitrariness of \( (\sigma, \tau) \) is restricted within (2.4).\(^2\) The ordinary velocity \( V_i \) of a certain element of the string is given as

\[
V_i(\sigma, t) = \frac{\partial x_i(\sigma, t)}{\partial t} = \left( \frac{\partial x_i(\sigma, \tau)}{\partial \tau} \right) \left| \frac{\partial x_0(\sigma, \tau)}{\partial \tau} \right|.
\]

Then the causality requires that \(|V| \leq 1\):

\[
\left( \frac{\partial x(\sigma, t)}{\partial t} \right)^2 \leq 1, \text{ i.e., } G^{00} = \left( \frac{\partial x^0}{\partial \tau} \right)^2 \leq 0 \text{ everywhere.}
\]

(We employ the unit system where \( c=1 \).) The condition (2.8) is invariant under (2.4). Now, if the condition (2.8) is slightly more stringent such that \( G^{00} < 0 \), then we have \( (\partial x_0/\partial \tau)^2 > 0 \) everywhere, and by reason of continuity, \( \partial x_0/\partial \tau \) must have a definite sign. That is, \( \partial x_0/\partial \tau \) is either forward timelike everywhere, or backward timelike everywhere. The proper-time element for each parton is \( (-G^{00})^{1/2} \, d\tau \), and the 4-velocity is

\[
U_\mu = \frac{\partial x_\mu}{\partial \tau} \sqrt{-G^{00}}.
\]

Conversely we have

\[
\frac{\partial x_0}{\partial \tau} = \frac{\sqrt{-G^{00}}}{\sqrt{1-V^2}}, \quad \frac{\partial x_i}{\partial \tau} = \frac{\sqrt{-G^{00}} V_i}{\sqrt{1-V^2}},
\]

(2.9)\(^3\)

Next the ordinary 3-dimensional radius vector going from the position of an element \( \sigma \) to that of \( \sigma + d\sigma \) of the string, viewed at time \( t \) by a Lorentz observer is

\[
\left( \frac{\partial x_i(\sigma, t)}{\partial \sigma} \right) d\sigma = W_i(\sigma, t) d\sigma = \left( \frac{\partial x_i(\sigma, \tau)}{\partial \sigma} - V_i \frac{\partial x_0(\sigma, \tau)}{\partial \sigma} \right) d\sigma.
\]

(2.10)

This is rewritten as

\(^1\) Strictly speaking we have \( \partial x_0/\partial \tau = \pm \sqrt{-G^{00}} / \sqrt{1-V^2} \), but because \( \partial x_0/\partial \tau \) has definite sign everywhere, we have written as in (2.9), taking the positive case.
\[ W_t = \frac{\partial (x_t, x_\sigma)}{\partial (\sigma, \tau)} \left| \frac{\partial x_\sigma}{\partial \tau} \right| = \frac{x_{t,1} x_{\sigma,0} - x_{t,0} x_{\sigma,1}}{x_{\sigma,0}}, \quad (2.10') \]

where we employ the notation

\[ x_{\mu,0} = \frac{\partial x_\mu(\sigma, \tau)}{\partial \tau}, \quad x_{\mu,1} = \frac{\partial x_\mu(\sigma, \tau)}{\partial \sigma}. \]

Corresponding to (2.6) we have \((dx)^2 = W^2 d\sigma^2 + V^2 dt^2 + 2V W d\sigma dt\). In particular the length along the string from the end \(\sigma = 0\) to the point \(\sigma\), viewed by Lorentz observer, is

\[ l(\sigma, t) = \int_{\sigma}^{\sigma'} \sqrt{W^2(\sigma, t)} d\sigma. \quad (2.11) \]

In the present realistic viewpoint \(V_t\) (whence \(U_{st}\)) and \(W_t\) are observables and are indeed left invariant under (2.5). We have the relations

\[ \frac{\partial x_\sigma(\sigma, \tau)}{\partial \sigma} = \frac{V W}{1 - V^2} \frac{G_0^2}{\sqrt{-G_{00}(1 - V^2)}}, \quad W^2 = \frac{D_0}{G_{00}} - \left(\frac{V W}{1 - V^2}\right)^2. \quad (2.12a, b) \]

We define the "normal velocity" \(V^1\) orthogonal to the string:

\[ V^1_t = \zeta_{tk} V_k, \quad (\zeta_{tk} = \delta_{tk} - W_t W_k/W^2, \quad \zeta_{tk} W_k = 0) \quad (2.13) \]

Then we have the relations

\[ (V^1)^2 = V^2 - \left(\frac{V W}{W^2}\right)^2, \quad D_0 = -\left(\frac{\partial x_\sigma}{\partial \tau}\right)^2 W^2 (1 - (V^1)^2). \quad (2.14a, b) \]

Thus we see the following: (i) The condition (2.3) means that \(|V^1| \leq 1\), (ii) condition (2.8) is stronger than (2.3), and (iii) if (2.8) holds we have always

\[ G^{ii} \geq (G^{00})^2/G_{00}. \quad ((G^{10})^2/G_{00} \leq 0) \quad (2.15) \]

Now any physically realizable motion of the string has to satisfy a certain equation of motion and boundary condition on \(x_\mu(\sigma, \tau)\), but they are physically meaningful only if they are covariant under (2.5), and should be derived from an action integral which is invariant under (2.5) (as well as under the Poincaré group). We find that for the free case such an action is restricted (in so far as we discard extremely complicated possibility) to the form

\[ A = \int_{\tau_0}^{\tau_1} d\tau \int_{\sigma}^{\sigma'} L d\sigma, \quad L = -\kappa \sqrt{1 - D_0}, \quad \kappa = \sqrt{\mu_0 + \omega G^0}, \quad (2.16) \]

and we assume that \(\omega \geq 0\). Then, under the causality condition (2.8), we have \(D_0 = D_0 + \omega G^{00} \leq 0\). The above \(L\) contains two structure constants \(\kappa\) and \(\omega\) with dimensions \([ML^{-1}]\) and \([L^2]\), respectively. In place of them we may employ the constants

\[ \mu_0 = \kappa \sqrt{\omega}, \quad K = \kappa / \sqrt{\omega} \]
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with dimensions \([M]\) and \([ML^{-2}]\), respectively. They are related to the mass density of the string and the modulus of internal tension (cf. \(\S\) 4), and \(L\) is rewritten as

\[ L = -\mu_0 \sqrt{- (G^{00} + K\mu_0^{-1}D_0)}. \]  

\((2.16')\)

\((2.5)\) and \((2.16')\) indicate that the theory is natural generalization of the relativistic point mechanics, where the free equation of motion must be derived from the action integral \(A = -m_0 \sqrt{-(dx_\tau/d\tau)^2} \) which is invariant under arbitrary transformation \(\tau \rightarrow \tau' (\tau)\). Indeed the string Lagrangian \((2.16')\) consists, under the root sign, of the kinetic term \(-G^{00}\) (corresponding to \(- (dx_\tau/d\tau)^2\) in point mechanics) and the tension term \(- (K/\mu_0) D_0\) which is related to the area of the strip. As seen later the two structure constants determine the trajectory slope and the leading intercept by

\[ \alpha' = \gamma(\pi \kappa \epsilon)^{-1}, \quad \alpha(0) = -\pi \kappa \omega / (2\hbar). \]

\[ \text{[See Eq. (3.35).]} \]

We assume that the constant \(\kappa\) is a universal constant with the value \(\kappa \approx 0.4 \text{ gr/sec.}\) Though this \(\kappa\) does not occur in the fundamental equations in free case, it defines the scale of momentum and angular momentum.

Now the variational principle that \(\delta A = 0\) with respect to an arbitrary variation \(\delta x_\mu (\sigma, \tau)\) vanishing at initial \(\tau_0\) and at final \(\tau_1\), yields the Euler equation and the boundary condition

\[ \frac{\partial p_\mu}{\partial \tau} = \frac{\partial S_\mu}{\partial \sigma}, \quad S_\mu|_{\tau_0, \tau_1} = 0, \]  

\[ \text{ }(2.17a, b) \]

where

\[ p_\mu = \frac{\partial L}{\partial \dot{x}_\mu}, \quad \frac{\partial L}{\partial x_\mu}, \quad \frac{\partial L}{\partial x_\mu} = \kappa (G^{11} + \omega) x_{\mu 0} - G^{10} x_{\mu 1}, \]  

\[ S_\mu = -\frac{\partial L}{\partial x_\mu} \frac{G^{10} x_{\mu 0} - G^{20} x_{\mu 1}}{\sqrt{-D_0}}. \]  

\[ \text{ }(2.18a, b) \]

Thus \((2.17b)\) means \(x_{\mu 1} = (G^{10}/G^{00}) x_{\mu 0}\) at the ends. There hold the identities

\[ S^\mu x_{\mu 0} = 0, \quad x_{\mu 0} p^\mu - L = 0. \]  

\[ \text{ }(2.19a, b) \]

We verify that \(S_\mu\) is spacelike everywhere by causality. From \((2.18)\) we obtain the converse relations

\[ \frac{\partial x_\mu}{\partial \tau} = \frac{S^\mu p_\mu - (pS) S_\mu}{\kappa \sqrt{-\Gamma_0}}, \quad \frac{\partial x_\mu}{\partial \sigma} = \frac{(pS) p_\mu - (p^2 + \mu_0^2) S_\mu}{\kappa \sqrt{-\Gamma_0}}, \]  

\[ \text{ }(2.20) \]

where \(\Gamma_0 = -\Gamma_0 + \mu_0^2 S^2, \) \(\Gamma_0 = \mu_0^2 S^2 + \kappa (pS)^2\). Also we have

\[ G^{00} = -\frac{S^\mu \Gamma_0}{\kappa \Gamma_0}, \quad G^{11} + \omega = -\frac{(p^2 + \mu_0^2) \Gamma_0}{\kappa \Gamma_0}, \quad G^{20} = -\frac{(pS) \Gamma_0}{\kappa \Gamma_0}, \]  

\[ \text{ }(2.21) \]

and \(D_0 = \Gamma_0^2 / (\kappa \Gamma_0)\). Note that \(\Gamma_0 \leq 0\) by causality. We can express \(D_0\) and \(S_\mu\)
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in terms of \( V, W \) and \( x_{0,\alpha} = \partial x_0 / \partial \tau \):

\[
D_\alpha = - (x_{0,\alpha})^2, \quad A_\alpha = (W^2 + \omega) (1 - V^2) + (VW)^2, \quad (2.22)
\]

\[
S_\alpha = \frac{e x_{0,\alpha} [(1 - V^2) W_\alpha + (VW) V_\alpha]}{\sqrt{A_\alpha}}, \quad S_\beta = \frac{e x_{0,\alpha} (VW)}{\sqrt{A_\alpha}}. \quad (2.23)
\]

Due to \((2.19a)\), Eq. \((2.17b)\) implies three conditions for each end. In fact it is reexpressed as

\[
W |_{0,\tau} = 0. \quad (2.24)
\]

Thus the parton density \textit{per unit length} of the string goes to infinity at the ends. The boundary condition \((2.17b)\) also means that at the ends we have \( x_{\mu,0} = (G^\mu / G^0) x_{\mu,0}, \; D_0 = 0, \; D_\alpha = \omega G^\alpha \), whence

\[
p_\nu(0, \tau) = \frac{\mu_0}{\sqrt{-G^0}} \frac{\partial x_\nu(0, \tau)}{\partial \tau} = \mu_0 U_\nu, \quad (p_\nu(0, \tau))^2 = -\mu_0^2, \quad (2.25)
\]

and the same relation at \( \sigma = \pi \). This means that, although \( p_\nu(\sigma, \tau) \) generally depends on the gauge, \( p_\nu(0, \tau) \) and \( p_\nu(\pi, \tau) \) are invariant under \((2.5)\) and always 4-vectors. They are observables and play important roles. (See §3.)

Equations \((2.17a, b)\) imply the 4-momentum conservation law which holds on the world strip \textit{locally}. Thus the 4-momentum of the system is given by the line integral

\[
P_\mu = \int_C (p_\nu d\sigma + S_\nu d\tau), \quad (2.26)
\]

where \( C \) is an arbitrary curve on the strip going from one end \( x_\mu(0, \tau) \) at any \( \tau \) to the other end \( x_\mu(\pi, \tau') \) at any \( \tau' \). \((2.26)\) does not depend on the choice of \( C \), because \( p d\sigma + S d\tau \) is a total differential due to \((2.17a)\) and because moreover we have \((2.17b)\). Taking any equal \( \tau \) curve on the strip for \( C \), we get

\[
P_\mu = \int_0^\pi p_\nu(\sigma, \tau) d\sigma, \quad dP_\mu / d\tau = 0. \quad (2.27)
\]

Similarly the angular momentum tensor is

\[
M_{\mu \nu} = \int_C (x_{\nu,\alpha} p_\alpha d\sigma + x_{\nu,\alpha} S_\alpha d\tau), \quad (2.28)
\]

which again does not depend on \( C \) and is written as

\[
M_{\mu \nu} = \int_0^\pi x_{\nu,\alpha}(\sigma, \tau) p_\alpha(\sigma, \tau) d\sigma, \quad dM_{\mu \nu} / d\tau = 0. \quad (2.29)
\]

\( P_\mu \) and \( M_{\mu \nu} \) are invariant under \((2.5)\) and are observables.\(^*\)

\( p_\nu(\sigma, \tau) \) is also the canonical conjugate to \( x_\mu(\sigma, \tau) \), but it is verified that

\(^*\) Moreover they are invariant under \((2.2)\).

\(^*\) Note that in an arbitrary gauge \( p_\nu(\sigma, \tau) \) is not necessarily Lorentz 4-vector except at the ends but \( P_\mu \) is always 4-vector.
there exists one local identity between $p_\mu$ and $\partial x_\mu / \partial \sigma$:
\[
\left( \frac{p}{\partial \sigma} \right)^2 + \frac{\mu_0}{K} (p^2 + \mu^2) + \mu_G \left( \frac{\partial x}{\partial \sigma} \right)^2 = 0,
\]
which reflects the covariance of the theory under (2.5).

Next we reduce (2.17a) as follows: We introduce the matrix $\widehat{G}$ and its inverse:
\[
\widehat{G} = \begin{pmatrix} G_{11} + \omega & G_{10} \\ G_{01} & G_{00} \end{pmatrix}, \quad \widehat{G}^{-1} = \frac{1}{D_0} \begin{pmatrix} G_{00} & -G_{10} \\ -G_{10} & G_{11} + \omega \end{pmatrix}.
\]
Then $D_0 = \det \widehat{G}$. Equation (2.17a) is reexpressed as
\[
\bar{q}_{\mu \nu} Q^\nu = 0,
\]
\[
Q_\nu = D_0 (\widehat{G}^{-1})_{\beta \gamma} \frac{\partial^2 x_\nu}{\partial \zeta_\beta \partial \zeta_\gamma} = (G_{11} + \omega) \frac{\partial^2 x_\nu}{\partial \tau^2} - 2G_{10} \frac{\partial^2 x_\nu}{\partial \tau \partial \sigma} + G_{00} \frac{\partial^2 x_\nu}{\partial \sigma^2},
\]
\[
\bar{q}_{\mu \nu} = g_{\mu \nu} - (\widehat{G}^{-1})_{\beta \gamma} \frac{\partial x_\mu}{\partial \zeta_\beta} \frac{\partial x_\nu}{\partial \zeta_\gamma}, \quad \left( \frac{\partial x^\mu}{\partial \tau}, \frac{\partial x^\nu}{\partial \tau} \right) \bar{q}_{\mu \nu} = 0.
\]
Unless $\omega = 0$ we can further prove that (2.31) is brought to the simpler form
\[
\bar{q}_{\mu \nu} Q^\nu = 0,
\]
\[
\bar{q}_{\mu \nu} = g_{\mu \nu} - \frac{\partial x_\mu}{\partial \tau} \frac{\partial x_\nu}{\partial \tau} \left| G_{00} \right| \left( \frac{\partial x^\mu}{\partial \tau}, \frac{\partial x^\nu}{\partial \tau} \right) = 0,
\]
\[
\bar{q}_{\mu \nu} = g_{\mu \nu} - \frac{\partial x_\mu}{\partial \tau} \frac{\partial x_\nu}{\partial \tau} \left| G_{00} \right| \left( \xi_{\mu \nu} = 0, \quad \xi_{\mu \nu} \xi^\nu = \xi_{\mu \lambda} \right)
\]
Clearly (2.32) is invariant under the "partial general transformation" (2.5) and contains three independent equations. This corresponds to the fact that each element of the string has three degrees of freedom. Equation of motion (2.32) is highly nonlinear but we can derive therefrom new physical conservation laws, aside from the conservation laws due to Poincaré invariance. We have
\[
\frac{\partial}{\partial \tau} \left( \frac{G_{10}}{\sqrt{-D_0}} \right) = \frac{\partial}{\partial \sigma} \left( \frac{G_{00}}{\sqrt{-D_0}} \right),
\]
\[
\frac{\partial}{\partial \tau} \left[ x_{\mu,1} x_{\nu,1} - \frac{G_{11} + \omega}{G_{00}} x_{\mu,\nu} \xi_{\nu,0} \right]
\]
\[
= \frac{\partial}{\partial \sigma} \left[ x_{\mu,0} x_{\nu,1} + x_{\mu,1} x_{\nu,0} - \frac{2G_{10}}{G_{00}} x_{\mu,0} x_{\nu,0} \right].
\]
These hold of course for the case $\omega \neq 0$, as they are derived from (2.32). On the other hand the relation
\[
\frac{\partial x_\mu}{\partial \tau} \frac{\partial x_\nu}{\partial \tau} = -S_\mu \frac{\partial x^\nu}{\partial \sigma},
\]
holds regardless of whether $\omega \neq 0$ or $\omega = 0$. If $\omega \neq 0$, (2.35) leads to (2.33). Equation (2.34) yields
\[ \int_0^\infty \left[ (G^\mu + \omega) U_\mu U_\nu + \frac{\partial x_\mu}{\partial \sigma} \frac{\partial x_\nu}{\partial \sigma} \right] d\sigma = \frac{1}{\kappa^2} \int_0^\infty \left( p_\mu p_\nu - \frac{p^2 + \mu^2}{S^2} S_\mu S_\nu \right) d\sigma = \text{const.} \quad (2.36) \]

We mention some limiting cases of this model. First the "geometric model limit" is given by \( \omega \to 0 \), where \( L \to -\kappa \sqrt{-D_0} \). (See II.) Another is the "tensionless limit", which is clearly given by \( K \to 0 \) with \( \mu_0 = \text{fixed} \neq 0 \), where \( L \to -\mu_0 \sqrt{-G^{00}} \). The opposite to this is the "local theory limit". Here \( \partial x_\mu(\sigma, \tau) / \partial \sigma = 0 \), so that we have \( p_\mu = \mu_0 x_\mu / \sqrt{-G^{00}}, S_\mu = 0 \), and the equation of motion reduces to that of point particle, as it should do. This corresponds to the limit \( K \to \infty \) with \( \mu_0 = \text{fixed} \neq 0 \). (See § 3.)

§ 3. Lorentz-covariant formalism and solutions

We now choose a convenient gauge in exploiting the invariance of (2.17) or (2.32) under (2.5). We have two particularly important gauges, the "Lorentz-covariant" one and the "Lorentz-noncovariant" one. In this section we consider the former, where we impose

\[ G^{00} = 0, \quad \text{i.e.,} \quad U^\mu(\partial x_\mu / \partial \sigma) = 0. \quad (3.1) \]

This means to choose \( \tau \) such that an equal \( \tau \) curve on the strip is everywhere orthogonal to world lines. Such \( \tau \) must now be a Lorentz scalar, and \( \sigma \) and \( \tau \) constitute orthogonal curvilinear coordinates on the strip such that \( (dx_\mu)^2 = G^{00} d\tau^2 + G^{11} d\sigma^2 \), and the two light-like directions lying on the strip is \( \sqrt{-G^{00}} d\tau \pm \sqrt{G^{11}} d\sigma = 0 \). Under (3.1), the inequality (2.15) is simplified to

\[ G^{\mu\nu} \geq 0 \quad \text{everywhere.} \quad (3.2) \]

Originally the transformation (2.5) implied that the \( \tau \)'s origin can be taken arbitrarily for each world line, but (3.2) now ensures that \( \tau \) is an "instant parameter" because an equal \( \tau \) curve on the strip is a space-like curve and also \( \partial x_0 / \partial \tau \) has definite sign. Owing to this fact, the definitions of the global quantities of the system, such as the geometrical center-of-mass \( X_\mu \), as integrals with respect to \( \sigma \) at equal \( \tau \) (see (3.9)) become physically suitable, and also \( p_\mu \) becomes everywhere timelike.

Also (3.1) is reexpressed as (see (2.12))

\[ \partial x_\mu(\sigma, \tau) / \partial \sigma = (VW) / (1 - V^2). \quad (3.3) \]

From (2.33) and (3.1) we get \( G^{\mu\nu} + \omega = -\varphi(\tau) G^{00} \), where \( \varphi(\tau) \) is an arbitrary function of \( \tau \), but satisfies \( \varphi(\tau) > 0 \) (in so far as \( \omega \geq 0 \)) in order to be consistent with the causality inequalities. Further Eqs. (2.32) and (2.17b) are simplified to \( \partial^2 x_\mu / \partial \sigma^2 = \varphi(\tau) \partial^2 x_\mu / \partial \tau^2 + 1/2 (d\varphi / d\tau) (\partial x_\mu / \partial \tau) \) and \( x\mu|_{0,\nu} = 0 \). Next we employ \( \bar{\tau} = \int \varphi(\tau)^{-1/2} d\tau \) in place of \( \tau \). Then we have \( G^{\mu\nu} + \omega = - (\partial x / \partial \bar{\tau})^2 \), and...
thus the matrix $\tilde{G}$ becomes diagonal and traceless. Owing to these relations the parameter $\tau$ is uniquely fixed (aside from $\tau \rightarrow \tau + \delta$) so that it must have a definite physical meaning. Indeed we have now $\left(\partial x_\mu/\partial \tau\right)_{t,\nu,-\omega} = -\omega$, so that $\tau = \sqrt{\omega^2}$ represents the proper time of the ends (unless $\omega = 0$). Henceforth in this section we write this $\tau$ simply as $\tau$. Thus finally the fundamental equations in this gauge consist of the d'Alembert equation and the open-end boundary condition

$$\frac{\partial^2 x_\mu}{\partial \tau^2} = \frac{\partial^2 x_\mu}{\partial \sigma^2}, \quad \frac{\partial x_\mu}{\partial \sigma} \mid _{t,\nu,-\omega} = 0, \quad (3 \cdot 4a, b)$$

and the constraint

$$G^\infty + G^{\mu} = -\omega. \quad (3 \cdot 5)$$

From (3.2) and (3.5) we have $G^\infty \leq -\omega$ so that $G^\infty < 0$ everywhere in so far as $\omega > 0$. Then $|V|$ is smaller than light velocity everywhere and $\partial x_\mu/\partial \tau$ has definite sign. The quantities $p_\mu$ and $S_\mu$ of (2.18) now become

$$p_\mu = \kappa \partial x_\mu/\partial \tau = \mu \partial x_\mu/\partial \tau, \quad S_\mu = \kappa \partial x_\mu/\partial \sigma. \quad (3 \cdot 6)$$

The remarkable feature of the present gauge is that it produces curious pseudo-symmetry between $\sigma$ and $\tau$ notwithstanding the asymmetry of the original Lagrangian (2.16) due to non-zero $\omega$. This pseudo-symmetry means that the fundamental equations, apart from the boundary condition, are invariant under the interchange $\sigma \rightarrow \tau, \tau \rightarrow \sigma$. Since $\sigma$ and $\tau$ have different domains, this "symmetry" has a local sense only. Another feature is that the equation of motion is originally nonlinear but, with the present constraint which itself is quadratic, becomes linear. Of course the superposition principle does not apply to physical solutions.

Let us now look at the Lorentz-covariant formalism derived above from a different standpoint, in dividing the theory into two steps: In the first step we consider the linear equations (3.4) alone ignoring the constraint. At this stage the theory has conformal invariance (see below), but of course (3.4a, b) are not sufficient to define a physical model because they do not ensure the "uniqueness of physical interpretation" nor causality. In the second step it is proved\(^a\) that to ensure these requirements we must impose the the condition (3.5), which works to break the conformal symmetry unless $\omega = 0$. [The results in the following apply to the $\omega = 0$ case also, where, however, caution must be payed as to interpretation (see II).] For a while we consider the first step and summarize some characteristic points.

(i) Equations (3.4) are derivable from the simpler action integral

$$A_1 = \int d\tau d\sigma L_1, \quad L_1 = \frac{k}{2} (G^\infty - G^{\mu\nu} - \omega), \quad (3 \cdot 7)$$

\(^a\)\text{Note that the condition (3.1) alone does not yet fix the gauge uniquely, because when (3.1) holds we equally have $(\partial x_\mu/\partial \tau)(\partial x_\nu/\partial \tau') = 0$ with the use of arbitrary function $\tau'(\tau)$.}
which is numerically equal, under the constraint, to the original \( L \). Then we have the corresponding Hamiltonian density \( H_i = (\partial x_i / \partial \tau)p_i - L_i = (\kappa/2)(G^0 + G^\mu_\nu + \omega) \), which gives the scalar Hamiltonian

\[
\mathcal{H} = \int_0^\tau H_i d\delta = A^0 + \frac{\pi \kappa}{2} \omega, \quad A^\nu = \frac{1}{2} \int_0^\tau \left[ \frac{1}{\kappa} p^0 + \kappa \left( \frac{\partial x^0}{\partial \delta} \right)^2 \right] d\delta. \tag{3.8}
\]

This allows the canonical formalism in which the role of \( t \) in the usual case is taken over by \( \tau \), because the equation of motion (3.4) is reproduced via \( \partial x^\mu / \partial \tau = \partial H_i / \partial p^\nu, \partial p^\nu / \partial \tau = - \partial H_i / \partial x^\mu \).

(ii) The definition of center-of-mass depends on the gauge, but

\[
X^\mu(\tau) = \frac{1}{\pi} \int_0^\tau x^\mu(\sigma, \tau) d\sigma \tag{3.9}
\]
defined in the present gauge is an unambiguous quantity, which we shall call the "geometrical center-of-mass". It satisfies

\[
\frac{dX^\mu}{d\tau} = \frac{p^\mu}{\pi \kappa}, \quad \frac{d^2X^\mu}{dt^2} = 0,
\]
so that \( m\tau/(\pi \kappa) = m\tau/\mu \) has the meaning of the proper-time of the center-of-mass. (This applies to the \( \omega = 0 \) case inclusive.)

(iii) The action integral \( A_1 \) and Eqs. (3.4a, b) are no longer invariant under (2.5) but have very rich symmetry properties, which are not necessarily shared by the original action integral \( A \). They are invariant under several "external transformations" acting on \( x^\mu \) as well as under several "internal transformations" acting on \( (\sigma, \tau) \). The translation and Lorentz groups, and also the external dilatation \( x^\mu(\sigma, \tau) \rightarrow \lambda x^\mu(\sigma, \tau) \), belong to the former. (Note, however, that under dilatation the action integral is not invariant.)

The internal transformation which leaves \( A_1 \), whence (3.4) invariant, consists of special linear transformations, which do not mix \( \sigma \) and \( \tau \), and the nonlinear conformal transformation, which mixes \( \sigma \) and \( \tau \) and does not belong to (2.4). The former consists of the internal scale transformation \( (\sigma' = \alpha \sigma, \tau' = \alpha \tau, \sigma_\theta' = 0, \sigma_\tau' = \alpha \tau'), \tau \)-displacement \( (\tau' = \tau + \delta) \), internal reflection \( (\sigma' = \pi - \sigma) \) and \( \tau \)-reversal. The transformation (2.4) is reduced to this smaller, special linear group,\(^{19}\) and the causality condition (3.2) is preserved under this transformation. By conformal transformation we mean the one which leaves the relation \( d\sigma^2 - d\tau^2 = 0 \) invariant satisfies \( [\partial \tau' / \partial \sigma]_H = 0 \).\(^9\) This is given in terms of an arbitrary periodic function \( f(\lambda) = \sum_n f_n e^{in\lambda} \) (\( f_n^* = f_{-n} \)), as\(^{19}\)

\[
\begin{align*}
\sigma'(\sigma, \tau) &= \sigma + \frac{1}{2} \left( f(\tau + \sigma) - f(\tau - \sigma) \right) = \sigma + i \sum_n f_n e^{in\sigma} \sin n\sigma, \\
\tau'(\sigma, \tau) &= \tau + \frac{1}{2} \left( f(\tau + \sigma) + f(\tau - \sigma) \right) = \tau + \sum_n f_n e^{in\sigma} \cos n\sigma.
\end{align*}
\tag{3.10}
\]

\(^9\) The latter relation is necessary to leave (3.4b) invariant.
The conformal symmetry implies that if we find a solution $x_\mu(\sigma, \tau)$ to the equation of motion and boundary condition (3.4), we obtain at once infinitely many other solutions $x_\mu'(\sigma, \tau)$ to be given by $x_\mu(\sigma^0(\sigma, \tau), \tau^0(\sigma, \tau)) = x_\mu'(\sigma, \tau)$. The variation which an infinitesimal conformal transformation ($f_\mu = \varepsilon_\mu$) induces on $x_\mu(\sigma, \tau)$ is

$$x^0(\sigma, \tau) - x(\sigma, \tau) = \sum_n \varepsilon_n e^{i\sigma r} \left( \cos n\sigma \frac{\partial x}{\partial \tau} + i \sin n\sigma \frac{\partial x}{\partial \sigma} \right).$$

We now mention the conservation laws related to the above symmetries. The conservation laws of 4-momentum and angular momentum tensor hold of course. They are (2.26) and (2.28), where $p_\mu$ and $S_\mu$ are now (3.6). There exist also an infinite number of conserved vectors\(^\text{(*)}\) $C_\mu = (1/2\pi)^{-1} e^{i\sigma r} \int \varepsilon_\mu(\cos r\sigma \partial x_\mu/ \partial \tau + i \sin r\sigma \partial x_\mu/ \partial \sigma) d\sigma$ ($r = \text{integer}$), each of which denotes the weighted amplitude of each normal mode except for $C_\mu = P_\mu/(\sqrt{2\pi})$.\(^\text{(*)}\)

The invariance of $A_1$ under the conformal group gives an infinite number of constants of motion

$$A' = e^{i\tau r} A'(\tau) = A'(0), \quad (r = \text{integer}) \quad (3.11)$$

where $A'(\tau) = \kappa \int_0^\infty \varepsilon_\mu(F \cos r\sigma + i G^{\mu\nu} \sin r\sigma) d\sigma$ and $F = (G^{\mu\nu} + G^{\nu\mu})/2$. $F$ and $G^{\mu\nu}$ satisfy $\partial F/\partial \tau = \partial G^{\mu\nu}/\partial \sigma$, $\partial G^{\mu\nu}/\partial \tau = \partial F/\partial \sigma$ and $G^{\mu\nu}|_{\sigma=0} = 0$. They are expressed conversely as

$$F = \frac{1}{\pi\kappa} \sum_{r=-\infty}^{\infty} A'(\tau) \cos r\sigma, \quad G^{\mu\nu} = \frac{-i}{\pi\kappa} \sum_{r=-\infty}^{\infty} A'(\tau) \sin r\sigma. \quad (3.12)$$

Since the conformal transformation concerns the internal parameters it commutes with the Poincaré group acting on $x_\mu$. Therefore $P_\mu$ and $M_{\mu\nu}$ are conformal invariant, while $X_\mu$ is not.

The constant of motion related to the invariance under internal scale transformation is

$$B = \kappa \int_0^\infty G^{\mu\nu}(\sigma, \tau) \delta d\sigma + \tau A' - \frac{\kappa}{2} \int_0^\tau \left[ \sigma G^{\mu\nu}(\sigma, \tau) \right]_{\sigma=\tau} d\tau. \quad (3.13)$$

This is reexpressed as $B = i \sum' [(1/\tau')/r] A'$.

Otherwise we have the conservation law

$$A_{\mu\nu} = \int_0^\tau \left( \frac{1}{\kappa^2} p_\mu p_\nu + \frac{\partial x_\mu}{\partial \sigma} \frac{\partial x_\nu}{\partial \sigma} \right) d\sigma = \text{const.} \quad (3.14)$$

which corresponds to (2.36).

The transformation properties of various quantities under the special linear transformations are easily obtained. In particular the directly observable quantities, such as $U_\mu, W_\mu d\sigma, P_\mu$ and $M_{\mu\nu}$, must all be invariant against the parameter

\(^\text{(*)}\) This conservation law may be regarded as the result of the invariance of (3.4) under the $(\sigma, \tau)$-dependent translation $x_\mu(\sigma, \tau) \rightarrow x_\mu(\sigma, \tau) + d_\mu e^{in\tau} \cos n\sigma$ ($d_\mu = \text{const vector}$).
transformation (2.4), which includes the special linear transformations and (2.5).

(iv) (3.4a, b) imply that

$$\frac{\partial x_\mu(\sigma, \tau)}{\partial \sigma} \pm \frac{\partial x_\mu(\sigma, \tau)}{\partial \tau} = \frac{\partial x_\mu(0, \tau \pm \sigma)}{\partial \tau}$$

and that \(\partial x_\mu(0, \tau) / \partial \tau\) are 2\(\pi\)-periodic functions of \(\tau\). The momentum density at the end \(p_\mu(0, \tau) = \mu_0 U_\mu\) equals \(\kappa \partial x_\mu(0, \tau) / \partial \tau\) in the present gauge. We have thus

$$p_\mu(\sigma, \tau) \pm \kappa \frac{\partial x_\mu(\sigma, \tau)}{\partial \sigma} = p_\mu(0, \tau \pm \sigma),$$

$$p_\mu(0, \tau + 2\pi) = p_\mu(0, \tau).$$

Also, with the use of \(C^\tau_\mu\) defined before, we have

$$p_\mu(0, \tau) = \sqrt{2\kappa} \sum_{r=-\infty}^{\infty} C^\tau_\mu e^{-ir\tau}. \tag{3.17}$$

Clearly, \(p_\mu(\sigma, \tau)\) and \(\partial x_\mu(\sigma, \tau) / \partial \sigma\), whence \(G^{\sigma\mu}\), are all periodic functions in \(\tau\) also.

\(u_\mu(\tau) = x_\mu(0, \tau)\) and \(p_\mu(0, \tau) = \kappa du_\mu(\tau) / d\tau\) describe the motion of the end. They are very useful quantities, because a motion of the end \(\sigma = 0\) is completely specifies a motion of the whole string including that of the other end \(\sigma = \pi\). (See below.)

Now, \(P_\mu\) and \(A\) were originally defined as the integrals of the densities over \(\sigma\) at any equal \(\tau\), but they can be regarded instead as the integral (or the average) of the densities over one period of \(\tau\) at any equal \(\sigma\), such that

$$P_\mu = \frac{1}{2} \int_{-\pi}^{\pi} p_\mu(\sigma, \tau) d\tau, \quad A^\mu = \frac{\kappa}{2} \int_{-\pi}^{\pi} F(\sigma, \tau) d\tau .$$

This is again a situation which reflects the "\(\sigma-\tau\) symmetry". Taking in particular \(\sigma = 0\), we get

$$P_\mu = \frac{1}{2} \int_{-\pi}^{\pi} p_\mu(0, \tau) d\tau , \quad A^\mu = \frac{1}{4\kappa} \int_{-\pi}^{\pi} (p_\mu(0, \tau))^2 d\tau . \tag{3.18a, b}$$

Further we have

$$A^\mu = \frac{1}{4\kappa} \int_{-\pi}^{\pi} (p_\mu(0, \tau))^2 e^{i\tau\tau} d\tau, \tag{3.19}$$

$$X_\mu(0) = \frac{1}{2\pi} \int_{-\pi}^{\pi} x_\mu(\sigma, \tau) d\tau ,$$

$$x_\mu(\sigma, \tau + 2\pi) - x_\mu(\sigma, \tau) = u_\mu(\tau + 2\pi) - u_\mu(\tau) = (2/\kappa) P_\mu = \text{const},$$

$$x_\mu(\sigma, \tau) \equiv x_\mu(0, \tau - \pi) + P_\mu / \kappa .$$

The transformations of \(u_\mu(\tau)\) and \(p_\mu(0, \tau)\) under the conformal transformation, which is now represented as
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\[ \tau' = \tau + f(\tau) = \tau + \sum_n f_n e^{ix_n}, \quad (3.20) \]

are easily determined as

\[ u'_\mu(\tau') = u_\mu(\tau), \quad p'_\mu(0, \tau') = p_\mu(0, \tau) \left( \frac{d\tau'}{d\tau} \right). \quad (3.21) \]

If we define the variation which a conformal transformation (3.20) induces on the functions \( u_\mu(\tau) \) by \( \tilde{u}_\mu(\tau) - u_\mu(\tau) \) with \( \tilde{u}_\mu(\tau) = u_\mu(\tau') \), this variation is

\[ \tilde{u}_\mu(\tau) - u_\mu(\tau) = f(\tau) p_\mu(0, \tau)/\kappa, \quad \tilde{p}_\mu(0, \tau) - p_\mu(0, \tau) = \frac{d}{d\tau} (f(\tau) p_\mu(0, \tau)) \]

for an infinitesimal conformal transformation \( f(\tau) = \text{ infinitesimal} \).

(v) If we introduce the relative coordinate and momentum, \( x_\mu(\sigma, \tau) = x_\mu(\sigma, \tau) - X_\mu(\tau) \) and \( p_\mu(\sigma, \tau) = p_\mu(\sigma, \tau) - P_\mu/\kappa \), then \( x_\mu(\sigma, \tau) \) is periodic in \( \tau \), and \( M_{\mu\nu} \) and \( A^\mu \) also split into a part due to the center-of-mass motion and a part due to the internal motion such that \( M_{\mu\nu} = X_{\mu\nu} + S_{\mu\nu}, A^\mu = [(1/2\kappa) (P_\mu)^2 + R]/\kappa \), where \( S_{\mu\nu} \) and \( R \), too, are constants of motion. (See Ref. 13.) The length of spin (i.e., the intrinsic angular momentum) is given (classically) by \( J = \sqrt{-(\omega_\mu)^2/(P_\mu)^2} \) with \( \omega_\mu = \frac{1}{2} \varepsilon_{\mu\nu\lambda} M^* P_\lambda = \frac{1}{2} \varepsilon_{\mu\nu\lambda} S^* P_\lambda \).

Now, in the second step we take account of the constraints, which are \( G^a = 0 \) and \( F = -\omega/2 \). It is clear that the constraint breaks the conformal invariance\(^*\) as well as the external dilatation symmetry, unless \( \omega = 0 \). The constraint is also represented as

\[ (du/d\tau)^2 = -\omega, \quad \text{i.e.,} \quad (p_\mu(0, \tau))^2 = -\mu_\sigma. \quad (3.22) \]

Now this implies that \( p_\mu(0, \tau)^2 > 0 \) in so far as \( \omega > 0 \), and therefore \( p_\mu(0, \tau) \) must have a definite sign by reason of continuity: either \( p_\mu(0, \tau) > 0 \) always, or \( p_\mu(0, \tau) < 0 \) always. From this fact and (3.22) we get \( G^a = (\partial x_\mu(\sigma, \tau)/\partial \tau)^2 = (p_\mu(0, \tau - \sigma) + p_\mu(0, \tau + \sigma))^2/(2\kappa)^2 \leq -\omega \), whence \( G^a \geq 0 \). Thus the causality inequalities are again verified. Indeed (3.22) means that the end moves causally and this guarantees that every point of the string also moves causally with three degrees of freedom. As remarked before, (3.22) means that \( \sqrt{\omega} \tau \) is the proper time of the ends. Moreover we can conversely prove that the condition that \( \tau \) should represent the proper time of the end is equivalent to the constraint (3.1), via the equation of motion and boundary condition. (3.22) also gives the relation \( (P_\mu)^2 \leq -\mu_\sigma \mu_\sigma \), so that no tachyonic motion occurs and the mass \( m \) satisfies \( m \geq \sqrt{\mu_\sigma \mu_\sigma} \). This result does not depend on the gauge and must be true in any gauge.

The constraint (3.22) is also expressed as

\[ A^\mu = -\mu\kappa \omega/2, \quad A^\mu = 0. \quad (r = \pm 1, \pm 2, \cdots) \quad (3.23a, b) \]

(3.23a) means that the squared mass is

\( \quad \text{by conformal transformation \( A^\mu \) alters its constant value.} \)
\[ m^2 = -(P_\mu)^2 = 2\kappa R + \pi^2 \mu_0^2. \] (3.24)

Also (3.23) implies that the conservation laws of \( A' \) are apparent ones, and \( A' \), in contrast to \( C'_\sigma \), are not those constants of motion that classify physical solutions.

The fact that under the constraints the parameters \( \sigma \) and \( \tau \) are essentially unique corresponds to the fact that they have definite physical meanings, as already explained. They have still slight arbitrariness within the special linear group, under which all fundamental equations are invariant. In any case, however, in the present gauge all \( \partial x_\mu/\partial \tau \) and \( \partial x_\mu/\partial \sigma \) can be expressed completely in terms of physical quantities \( V_i(\sigma, \tau) \) and \( W_i(\sigma, \tau) \). Indeed the constraint (3.5) yields the relation (see (2.12))

\[ G^{11} = -(G^{00} + \omega) = W^2 + (VW)^2/(1 - V^2), \] (3.25)

and thus all \( \partial x_\mu/\partial \tau \) and \( \partial x_\mu/\partial \sigma \) are expressed in \( V \) and \( W \) alone.\(^{10}\)

The present gauge is the most convenient for writing down the general solutions. In fact there are various ways to express the general solutions, each of which is useful.\(^{10}\)

(i) One can give it in the form of Cauchy problem. For that purpose we first define a scalar function \( D(\sigma, \tau) \). This is that solution of

\[ \frac{\partial^2 D}{\partial \tau^2} - \frac{\partial^2 D}{\partial \sigma^2} = 0, \quad \frac{\partial D}{\partial \sigma}|_{\sigma, \tau} = 0, \] (3.26)

which satisfies the initial conditions

\[ \frac{\partial D}{\partial \tau}|_{\tau = 0} = \delta(\sigma), \quad D(\sigma, 0) = 0. \]

Explicitly

\[ D(\sigma, \tau) = \frac{1}{4} \sum_{n=\infty}^{\infty} \left[ \delta(\tau - \sigma - 2n\pi) + \delta(\tau + \sigma - 2n\pi) \right] \]

\[ = \frac{1}{2} \delta(\tau) \sum_{n=\infty}^{\infty} \theta(\tau^2 - (\sigma + 2n\pi)^2), \] (3.27)\(^*\)

which is odd in \( \tau \) and vanishes in the region where \(|\tau| < \sigma\). Also we can write as \( D(\sigma, \tau) = (1/2) \sum_{n=-\infty}^{\infty} \left[ \theta(\sigma + \tau + 2n\pi) - \theta(\sigma - \tau + 2n\pi) \right] = (1/2\pi)(\tau + 2 \sum_{n=1}^{\infty} \sin n\pi \cos n\sigma/n). \) Incidentally we also give the even solution of (3.26) satisfying \([\partial D^{(1)}/\partial \tau]|_{\sigma=0} = 0:\)

\[ D^{(1)}(\sigma, \tau) = -\frac{1}{2\pi} \log [2(\cos \sigma - \cos \tau)] = \frac{1}{\pi} \sum_{n=-1}^{\infty} \frac{\cos n\sigma \cdot \cos n\tau}{n}, \] (3.28)

which is useful in interaction problem. Next we define \( J(\sigma, \sigma', \tau) = D(\sigma - \sigma', \tau) + D(\sigma + \sigma', \tau) \), which is even and symmetric in \( \sigma \) and \( \sigma' \) and has the properties that

\(^*\) Note that \( \frac{1}{2}[\delta(x-a) + \delta(x+a)] = \delta(x) \theta(a^2 - a^3). \)
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\[
\frac{\partial^2 \mathcal{A}(\sigma, \sigma', \tau)}{\partial \tau^2} \bigg|_{\tau=0} = \delta(\sigma - \sigma') + \delta(\sigma + \sigma') + \delta(2\pi - \sigma - \sigma'),
\]

(3.29)

\[
\mathcal{A}(\sigma, \sigma', \tau) = 0 \text{ for } |\tau| < |\sigma - \sigma'|.
\]

(3.30)

Then the solution to (3.4) is expressed as

\[
x_\mu(\sigma, \tau) = \int_0^\infty d\sigma' \mathcal{A}(\sigma, \sigma', \tau - \tau_0) \frac{\delta}{\delta \tau_0} x_\mu(\sigma', \tau_0).
\]

(3.31)

Due to the property of \(\mathcal{A}, x_\mu(\sigma', \tau_0)\) in the region \(|\sigma - \sigma'| > |\tau - \tau_0|\) on the strip does not contribute to \(x_\mu(\sigma, \tau)\). In the neighborhood of \(x_\mu(\sigma, \tau)\), the causal relation exists only for the region \(d\tau^2 - d\sigma^2 \geq 0\), which is the timelike region because we have \(x_\mu(\sigma + d\sigma, \tau + d\tau) - x_\mu(\sigma, \tau)^2 = G^{00}d\tau^2 + G^{11}d\sigma^2 = G^{00}(d\tau^2 - d\sigma^2) - \omega d\sigma^2 \leq 0\). This verifies the causality. The constraint (3.1) is not incorporated in (3.31).

However, if \(G^{00} = 0\) and \(\partial F/\partial \sigma = 0\) (whence \(\partial G^{10}/\partial \tau = 0\)) at \(\tau = \tau_0\), then \(G^{10} = 0\) at any \(\tau\), since \(G^{10}\) satisfies \(\partial^2 G^{10}/\partial \tau^2 = \partial^2 G^{10}/\partial \sigma^2\).

(ii) A simpler way is suggested from the "\(\theta - \tau\) symmetry"; namely we exchange the roles of \(\tau\) and \(\sigma\) in (i) and regard the problem as "Cauchy problem" with respect to \(\sigma\). Indeed the whole motion is determined once we are given the motion of the end \(x_\mu(0, \tau) = u_\mu(\tau)\), which must have the property that its derivative \(p_\mu(0, \tau) = \varepsilon \sigma\) satisfies the conditions (3.16) and (3.22). In terms of such \(u_\mu(\tau)\) the solution is given simply as \(x_\mu(\sigma, \tau) = \frac{1}{2} [u_\mu(\tau - \sigma) + u_\mu(\tau + \sigma)]\), which in fact covers all possible solutions. Since the motion of the end \(u_\mu(\tau)\) and \(p_\mu(0, \tau)\) dominates the motion of the whole string, all relevant quantities are expressed in terms of it, as already remarked.

(iii) Another form of general solution is obtained by noting (3.17). That is,

\[
x_\mu(\sigma, \tau) = X_\mu(0) + \sqrt{2} C_\mu \varepsilon + \sqrt{2} i \sum_{r=\infty}^{\infty} C_\mu r e^{-i\tau r} \cdot \frac{\cos r \sigma}{r}. \tag{3.32}
\]

This form is also clear from the fact that the complete set of solutions to (3.4) is given by \(X_\mu = X_\mu(0) + \sqrt{2} C_\mu \varepsilon\), and

\[
x_\mu^{(r, \alpha)}(\sigma, \tau) = \varepsilon_\mu \varepsilon_{\alpha} e^{-i\tau r} \cdot \frac{\cos r \sigma}{r}, \quad (r = \pm 1, \pm 2, \ldots; \alpha = 1, 2, 3, 0)
\]

where \(\varepsilon_\mu\) are vierbein vectors satisfying \(\varepsilon_\mu \varepsilon_{\alpha} = 0\), and imply the polarization vectors. In (3.32) \(C_\mu r = C_\mu^{(r)} \varepsilon_\mu\) with \(C_\mu^{(r)} = C_\mu^{(-r)}\) arbitrary. The inner product of two arbitrary solutions \(x_\mu^{(1)}(\sigma, \tau)\) and \(x_\mu^{(2)}(\sigma, \tau)\) is defined by

\[
(x^{(1)} \cdot x^{(2)}) = \int_0^\infty d\sigma x_\mu^{(1)*}(\sigma, \tau) \frac{\delta}{\delta \tau} x_\mu^{(2)}(\sigma, \tau),
\]

and the eigensolutions are mutually orthogonal: \((\varepsilon^{(r, \alpha)} \cdot \varepsilon^{(r', \alpha')}) = (\pi/\tau) \delta_{rr'} \delta_{\alpha\alpha'}\). Corresponding to (3.23), \(C_\mu r\) must satisfy the constraints

\[
\frac{1}{2} (C_\mu^2 + \sum_{n=1}^{\infty} C_\mu C_{\mu} = -\frac{\omega}{4}; \quad \sum_{n=1}^{\infty} C^* C_{\mu}^{(-n)} = 0. \quad (r \neq 0) \tag{3.33}
\]
Constants of motion are reexpressed as
\[ \Lambda' = \pi \kappa \sum_n C^n C'^n, \]
\[ R = 2\pi \kappa \sum_{n=-\infty}^{\infty} C_n C_n, \]
\[ S_{\nu\nu} = -2\pi \kappa \sum_{n=-\infty}^{\infty} C_{n}^{*} C_{n}/\nu, \]
\[ \Lambda_{\nu\nu} = 2\pi \sum_{n=-\infty}^{\infty} C_{n}^{-\nu} C_{n}^{*}. \]

(iv) Another obvious way is to start from the expansion

\[ x_{\nu}(\sigma, \tau) = \sum_{n=-\infty}^{\infty} x_{\nu}^{n}(\tau) \cos \nu r, \quad (x^{-\nu} = x^{\nu}) \]
\[ P_{\nu}(\sigma, \tau) = \frac{1}{2\pi} \sum_{n=-\infty}^{\infty} P_{\nu}^{n}(\tau) \cos \nu r, \quad (p^{-\nu} = p^{\nu}) \]

where \( x_{\nu}^{n}(\tau) \) satisfies \( d^2 x_{\nu}^{n}/d\tau^2 + \nu^2 x_{\nu}^{n} = 0 \), while \( p_{\nu}^{n}(\tau) = 2\pi \kappa \cdot dx_{\nu}^{n}(\tau)/d\tau. \) In particular \( x_{\nu}^{0}(\tau) = X_{\nu}(\tau), p_{\nu}^{0} = 2\pi \kappa \). If we then define

\[ C_{\nu}^{\tau}(\tau) = \frac{1}{\sqrt{2}} \left( \frac{dx_{\nu}^{\tau}(\tau)}{d\tau} - i\nu x_{\nu}^{\tau}(\tau) \right), \quad C_{\nu}^{-\tau}(\tau) = C_{\nu}^{\tau}(\tau)^* \]

the solution is \( C_{\nu}^{\tau}(\tau) = C_{\nu}^{\tau} e^{-i\nu \tau} \), where \( C_{\nu}^{\tau} \)

are the same constants as given before.

In the \( m^2 Jal \) plot the classical solutions are distributed continuously in the hatched region in Fig. 1. The solutions where only the first mode is excited besides \( C_{\nu}^{0} \) are simple and physically important.\(^a\)

They are specified by the complex vector \( C_{\nu}^{1} \)

which is restricted by

\[ P^{\nu} C_{\nu}^{1} = 0, \quad (C_{\nu}^{1})^{*} = 0. \quad (3.34a,b) \]

They give the solutions lying on the leading trajectory, which is given by

\[ J = \frac{R}{\pi} = \frac{m^2 \nu \kappa}{2\pi \kappa} - \frac{\nu \kappa \omega}{2}. \quad (3.35) \]

By eliminating \( \tau \) we see that the motion of the string for this solution is a rigid rotation and the string has constant length

\[ l = 2 \sqrt{\frac{2J}{\pi \kappa}} = 2 \sqrt{\frac{m^2 - \nu^2 \mu_0^2}{\mu_0 K}}, \quad (3.36)** \]

The more stretched the string is, the more massive it is. The string sweeps a world strip of skrew-like form. In the ground state, however, the string shrinks to a single point.\(^**\)

This means that our string has just vanishing equilibrium length.

\(^a\) If one picks out these solutions alone, the theory is essentially similar to the case of bilocal modelll, where the states on the daughter trajectories have to be suppressed by the condition \((3.34b)\).

\(^**\) The \( \nu \)-value mentioned in § 2 corresponds to \( \alpha' \sim 1 \) Bev\(^2\), and the above \( l \) means the extension of the order of nucleon Compton wave-length. If we assume that the dimensionless constant \( \pi^2 \kappa \omega/\hbar \) is of order 1, then we have \( \omega \sim 1.6 \times 10^{33} \text{cm}^2, K \sim 5 \times 10^{25} \text{gr/sec}^2 = 330 \text{gr/cm}^2, \mu_0 \sim 5 \times 10^{-25} \text{gr}. \)

\(^**\) In quantum mechanics, however, the spatial extension of the ground state is not essentially different from that of the excited states due to quantum fluctuation.
We may consider the limiting cases. In the tensionless limit \((K \to 0, \mu_\omega = \text{fixed} \neq 0)\) the leading trajectory becomes steeper to approach the vertical line (in Fig. 1), with the ground state at \((m^2 = \pi^2 \mu_\omega^3, J = 0)\) fixed. Then in a state on the leading trajectory, \(\bar{T}\) goes to \(\infty\). The opposite is the local theory limit, \(K \to \infty\) with \(\mu_\omega = \text{fixed} \neq 0\), where the trajectory becomes flat and \(\bar{T} \to 0\). Another limit is \(\omega \to 0\), where the leading trajectory starts from the origin due to the external dilatation invariance. If \(x_\sigma(\sigma, \tau)\) is a solution at \((m^2, J)\) then \(\lambda x_\sigma(\sigma, \tau)\) is a solution at \((\lambda^2 m^2, \lambda^2 J)\).

**§ 4. Non-covariant formalism**

The theory of realistic string model can also be represented in a Lorentz non-covariant formalism. This is obtained by either of the following methods:

(i) Since the original theory is covariant under arbitrary transformation \((2 \cdot 5)\), we can choose \(\tau'(\sigma, \tau)\) such that it equals \(x_\sigma(\sigma, \tau)\) in the Lorentz frame under consideration, namely

\[
\tau'(\sigma, \tau) = x_\sigma(\sigma, \tau) = t . \tag{4 \cdot 1}
\]

This fixes the gauge completely. Now \(x_\sigma(\sigma, \tau)\) becomes \(x_\sigma(\sigma, \tau') = x_\sigma(\sigma, t)\), and further we have

\[
\begin{align*}
\frac{\partial x_i(\sigma, \tau')}{\partial \tau'} &= \frac{\partial x_i(\sigma, t)}{\partial t} = V_i, & \frac{\partial x_i(\sigma, \tau')}{\partial \sigma} &= 1, \\
\frac{\partial x_i(\sigma, \tau')}{\partial \sigma} &= \frac{\partial x_i(\sigma, t)}{\partial \sigma} = W_i, & \frac{\partial x_i(\sigma, \tau')}{\partial \sigma} &= 0 .
\end{align*} \tag{4 \cdot 2}
\]

Thus in the present gauge \(G^{ab}\) becomes \(\bar{C}^{ab}\):

\[
\bar{C}^{\theta \theta} = -(1 - V^2), \quad \bar{C}^{ii} = W^i, \quad \bar{C}^{10} = V W , \tag{4 \cdot 3} \]

and \(D_{\omega}\) becomes \(-D_{\omega}\) (see Eq. (2 \cdot 22)), so that the invariant action integral (2 \cdot 12) is expressed as

\[
A = - \kappa \int_{t_1}^{t_2} dt \int_{\sigma} d\sigma \sqrt{J_{\omega}} . \tag{4 \cdot 4}
\]

By (4 \cdot 2) and (4 \cdot 3) we see at once that the equation of motion (2 \cdot 32) for the case \(\omega \neq 0\) goes over in the present gauge to the third-order equation

\[
(W^2 + \omega) \frac{\partial^3 x_i}{\partial t^3} - 2V W \frac{\partial^2 x_i}{\partial t \partial \sigma} - (1 - V^2) \frac{\partial^2 x_i}{\partial \sigma^2} = 0 , \tag{4 \cdot 5}
\]

while the boundary condition (2 \cdot 17b) is (2 \cdot 24), i.e.,

\[
W_i \big|_{\sigma = 0, \tau} = 0 . \tag{4 \cdot 6}
\]

The fundamental equations (4 \cdot 5) and (4 \cdot 6) in this gauge are Lorentz non-covariant.

\*\* Note that in this gauge \(\bar{C}^{\theta \theta} \leq 0, \bar{C}^{ii} \geq 0\) again, but \(\bar{C}^{10}\) is generally non-zero.
but free from constraint.

If we denote \( p_\mu \) in the present gauge as \( \Pi_\mu \), the 4-momentum of the system is

\[
P_\mu = \int_0^\tau \Pi_\mu(\sigma, t) \, d\sigma \quad \text{with} \quad \frac{dP_\mu}{dt} = 0,
\]

so that \( \Pi_t \) and \( \Pi_0 \) are the momentum and energy densities per unit \( \sigma \) (i.e., per parton) viewed by a Lorentz observer. From (2.18) and (4.2) they are

\[
\Pi_t(\sigma, t) = \frac{\kappa}{\sqrt{J_\sigma}} \left[ (W^2 + \omega) V_t - (VW) W_t \right] = \kappa \frac{\partial (-\sqrt{J_\sigma})}{\partial V_t},
\]

\[
\Pi_0(\sigma, t) = \kappa (W^2 + \omega) / \sqrt{J_\sigma}.
\]

\( \Pi_t \) is also the momentum variable canonically conjugate to \( x_i(\sigma, t) \), and \( \Pi \) is generally not parallel to \( V \). Also note that \( \Pi_0 \geq 0 \) everywhere by virtue of \( |V| \leq 1 \) and \( \omega \geq 0 \), which mean causality. Generally \( \Pi_\mu \) does not mean a 4-vector. However, as we have remarked in \( \S \, 2 \), \( \Pi_\mu \) at each end is gauge-independent and 4-vector such that \( \Pi_\mu = \mu U_\mu \) and equals \( p_\mu(0, \tau) \) (or \( p_\mu(\tau, \tau) \)) in the Lorentz-covariant gauge.

Similarly, \( S_\mu \) of (2.18b) now becomes

\[
\bar{S}_t(\sigma, t) = \frac{\kappa}{\sqrt{J_\sigma}} \left[ (1 - V^2) W_t + (VW) V_t \right] = \kappa \frac{\partial \sqrt{J_\sigma}}{\partial W_t},
\]

\[
\bar{S}_0(\sigma, t) = \kappa (VW) / \sqrt{J_\sigma}.
\]

Equation (2.17a) is expressed as

\[
\partial \Pi_t / \partial t = \partial \bar{S}_t / \partial \sigma, \quad \partial \Pi_0 / \partial t = \partial \bar{S}_0 / \partial \sigma
\]

in this gauge. The former is just the Euler equation resulting from the variational principle on (4.4), and we can prove that (4.5) can also be derived from this Euler equation in so far as \( \omega \neq 0 \). The relations (2.19) and (2.30) now take the forms

\[
\bar{S}_t = V \bar{S}, \quad \Pi_0 = \Pi V + \kappa \sqrt{J_\sigma},
\]

\[
\Pi_0 = \Pi^2 + \kappa^2 W^2 + \frac{1}{\omega} (\Pi W)^2 + \kappa^2 \omega. \quad (\omega \neq 0)
\]

Also we note the relations \( \Pi = \Pi_0 V - \bar{S}_0 W \), and

\[
V = [\Pi + (\Pi W) W / \omega] / \Pi_0, \quad (\omega \neq 0)
\]

\[
\Pi \bar{S} = \kappa^2 W + \bar{S}_0 \Pi, \quad [\partial \Pi / \partial \sigma]_{\mu, \tau} = 0,
\]

\[
\Pi W = \omega \bar{S}_0, \quad \bar{S}_0 / \Pi_0 = (VW) / (W^2 + \omega).
\]

In this gauge angular-momentum tensor is written as*\(^8\)

*\(^8\) Again \( \mu, \nu \) in \( M_{\mu \nu} \) are generally not 4-vector indices.
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\[ M_{\nu} = \int_{0}^{\pi} \mathcal{M}_{\nu}(\sigma, t) d\sigma, \]
\[ \mathcal{M}_{ij} = x_i d_i t, \quad \mathcal{M}_{i0} = x_i d_i t. \]  

(4.14)

\( \mathcal{M}_{ij} \) means the angular-momentum density per parton in the Lorentz frame. We can directly confirm the conservation law \( dM_{ij}/dt = dM_{i0}/dt = 0 \), which implies that the theory is Lorentz-invariant, even though the equation of motion is non-covariant. The "physical center-of-mass" should be defined by

\[ \bar{X}_i(t) = \int_{0}^{\pi} x_i(\sigma, t) \Pi_0(\sigma, t) d\sigma / P_i, \]  

(4.15)

which conforms with the usual definition of center-of-mass of a relativistic extended system.\(^{17} \) By (4.14), Eq. (4.15) is reexpressed as

\[ \bar{X}_i = (P_i / P_0) t + M_{i0} / P_0, \]

which is also the usual relation.

Aside from the above conservation laws due to Poincaré invariance there hold several conservation relations. They are derived from the equation of motion (4.5) and thus hold for the case \( \omega \neq 0 \). They are

\[ \frac{\partial \bar{S}_s}{\partial t} = \frac{\partial}{\partial t} \left( V W \right) = - \kappa \frac{\partial}{\partial \sigma} \left( \frac{1 - V^2}{\sqrt{A_s}} \right), \]  

(4.16)

\[ \frac{\partial}{\partial t} \left( \frac{W^2 + \omega}{1 - V^2} \right) = 2 \frac{\partial}{\partial \sigma} \left( \frac{W V}{1 - V^2} \right), \quad 2 \frac{\partial}{\partial t} \left( \frac{W V}{1 - V^2} \right) = \frac{\partial}{\partial \sigma} \left( \frac{1 - V^2}{W^2 + \omega} \right), \]  

(4.17a, b)

\[ \frac{\partial}{\partial t} \left( \frac{(W^2 + \omega) V_i}{1 - V^2} \right) = \frac{\partial}{\partial \sigma} \left( W_i + \frac{2(W V) V_i}{1 - V^2} \right). \]  

(4.18)

In fact (4.16), (4.17a) and (4.18) are nothing but the equations (2.33) and (2.34) in the present gauge. (4.17a) leads to

\[ \int_{0}^{\pi} \left[ \frac{(W^2 + \omega)}{1 - V^2} \right] d\sigma = \text{const}, \]  

(4.19)

while (4.18) is equivalent to the equation of motion (4.5), and implies the conservation law \( \int_{0}^{\pi} \left[ \frac{(W^2 + \omega) V_i}{1 - V^2} \right] d\sigma = \text{const}. \)

(ii) The second method to derive the non-covariant formalism is to start from the Lorentz covariant formalism given in § 3 and eliminate therefrom \( x_0(\sigma, \tau) \) and \( \tau \) with the aid of the constraint. First we derive the equation of motion for \( x_i(\sigma, t) \) [see(2.6)] from the original covariant equation (3.4a), to get

\[ \left( \frac{\partial}{\partial \sigma} \frac{\partial x_0(\sigma, \tau)}{\partial \tau} \right)^2 - \left( \frac{\partial}{\partial \sigma} \frac{\partial x_0(\sigma, \tau)}{\partial \tau} \right)^2 \frac{\partial^2 x_i(\sigma, t)}{\partial \sigma^2} + \frac{\partial^2 x_i(\sigma, t)}{\partial \sigma^2} + 2 \frac{\partial x_0(\sigma, \tau)}{\partial \sigma} \frac{\partial x_i(\sigma, t)}{\partial \sigma} = 0. \]

On the other hand, by virtue of the constraints, \( \partial x_0(\sigma, \tau) / \partial \tau \) and \( \partial x_0(\sigma, \tau) / \partial \sigma \) were expressed as in (2.9) and (3.3). Only by inserting them into the above equation, \( \tau \) and \( x_0(\sigma, \tau) \) can be completely eliminated and we recover the non-
linear equation of motion (4.5). Next in Eq. (2.26) with the use of (3.6) for $p_\sigma$ we now choose the curve $C$ to be $t=x_\sigma(r)=$ const, so that $\partial r = -((\partial x_\sigma/\partial \sigma)/
abla x_\sigma)\, dr$, and then we get $P_\sigma=\int_0^\infty \Pi_\sigma(\sigma, t)\, d\sigma$ with $\Pi_\sigma=\kappa \{ (\partial x_\sigma/\partial t) - (\partial x_\sigma/\partial \sigma) \cdot (\partial x_\sigma/\partial \sigma)/(\partial x_\sigma/\partial t) \}$, which is rewritten as (4.8), via (2.9), (3.3), etc. Similarly if we further introduce $\tilde{S}_\sigma=(\partial x_\sigma/\partial \sigma)/(\partial x_\sigma/\partial t)$, this is rewritten as (4.9). Now it is proved that from (4.5) we can derive (4.10b) and Eq. (4.16), and then furthermore (4.10a). In this method, we have derived the Euler equation (4.10a) from (4.5), but as already stated, we can conversely derive (4.9) from the Euler equation provided that $\omega \neq 0$, so that both are equivalent unless $\omega = 0$. This fact accords with the result of (i).

The transition from the covariant to the non-covariant formalisms in the second method has been effected by exploiting the conformal symmetry of the equation of motion and boundary condition of the covariant formalism. In the case $\omega >0$ this process consumes the conformal symmetry. Of course the Poincaré invariance and the invariance under internal scale transformation ($\sigma \rightarrow a \sigma$) persist. Besides them the equation of motion (4.5) has a certain formal symmetry between $\sigma$ and $t$, just as the covariant equation of motion had the pseudo-symmetry between $\sigma$ and $\tau$. To make this explicit we replace $\sigma$ by $s=\sqrt{\omega} \sigma$ running over $[0, \pi \sqrt{\omega}]$. Then (4.5) becomes

$$
(1+\tilde{W}^2) \frac{\partial^2 x_i}{\partial t^2} - 2(\tilde{V} \tilde{W}) \frac{\partial^2 x_i}{\partial t \partial s} - (1-V^2) \frac{\partial^2 x_i}{\partial s^2} = 0,
$$

where $\tilde{W}_i(s, t) = \partial x_i/\partial s$. This equation is formally invariant under $s \rightarrow s' = it$, $t \rightarrow t' = -is$, though the boundary condition is not invariant.

In the present gauge the theory is put in the standard Hamiltonian formalism. Indeed from (4.11) the Hamiltonian density is

$$
H=\Pi_0 = \left[ \Pi^2 + \mu_\sigma K \left( \frac{\partial x(\sigma, t)}{\partial \sigma} \right)^2 + \frac{K}{\mu_\sigma} \left( \Pi \frac{\partial x(\sigma, t)}{\partial \sigma} \right)^2 + \mu_0^2 \right]^{1/2}
$$

unless $\omega = 0$, and the Hamiltonian $H=P_\sigma=\int_0^\infty \Pi \, d\sigma$ reproduces the equation of motion via $V_t = \partial H/\partial \Pi$, $\partial^2 H/\partial \Pi^2 = -\partial^2 H/\partial x_i$, and there is no constraint.

Thus in the Hamiltonian formalism the main difference of our relativistic theory from the non-relativistic one is the occurrence of the square-root in the Hamiltonian density, and this is natural as relativization procedure. We also remark that in the case $\omega >0$ we have $|V| <1$ everywhere and we can consider the non-relativistic and weak-deformation limit, $|V| \ll 1$ and $|\partial x/\partial \sigma| \ll \sqrt{\omega}$. Then we have

$$
\Pi_0 \approx \mu_\sigma + \frac{\mu_0}{2} V^2 + \frac{K}{2} \tilde{W}^2, \quad \Pi_t \approx \Pi_0 V_t, \quad \tilde{S}_t \approx K W_t.
$$


*In the case $\omega = 0$, the non-covariant formalism retains the symmetry against $\sigma \rightarrow \sigma + \sum \beta_\nu(t) \chi \sin n \sigma$. (See II.)
These essentially coincide with the forms of non-relativistic string theory, and \( \mu_4 \) and \( K \) represent the mass density and the modulus of tension per parton.

Finally we remark that we can also expand \( x_i(\sigma, t) \) and \( \Pi_i(\sigma, t) \) in the form

\[
x_i(\sigma, t) = \sum_{n=-\infty}^{\infty} x_i^n(t) \cos n\sigma, \quad \Pi_i(\sigma, t) = \frac{1}{2\pi} \sum_{n=-\infty}^{\infty} \Pi_i^n(t) \cos n\sigma
\]

\[
(x_i^{-n} = x_i^n, \Pi_i^{-n} = \Pi_i^n)
\]

by virtue of the boundary conditions (see (4.13)). Then we introduce \( A_i^+(t) \) by

\[
A_i^+(t) = \frac{(2\pi \kappa)^{-1} \Pi_i^+(t) - i(2\pi \kappa)^{1/2} x_i^+(t)}{\sqrt{2}}
\]

\[
= \frac{2}{\sqrt{\pi \kappa}} \int_0^\pi d\sigma \left[ \Pi_i(\sigma, t) \frac{\cos r\sigma + i\sqrt{\kappa} \frac{\partial x_i}{\partial \sigma} \sin r\sigma}{\sqrt{\kappa}} \right].
\]

Then \( A_i^- = A_i^+ \), and we have \( P_i = \Pi_i^+ / 2 = \sqrt{\pi \kappa} A_i^0 \), which is time-independent. Also

\[
x_i(\sigma, t) = x_i^0(t) + \frac{i}{\sqrt{\pi \kappa}} \sum_{n=+\infty}^{\infty} \frac{A_i^+(t) \cos n\sigma}{n} r, \quad \Pi_i(\sigma, t) = \frac{\sqrt{\pi}}{\kappa} \sum_{n=-\infty}^{\infty} A_i^+(t) \cos n\sigma.
\]

In the general case \( A_i^+(t) \) obeys a non-linear (third-order) equation which is a differential equation of second degree in \( t \).

Though it is difficult to obtain general solutions for the fundamental equations in this gauge, solutions with purely transversal internal movement are obtained directly in this gauge (see II):

References

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