Ordinary Time Stochastic Quantization of Bosonic String
and Its Supersymmetric Effective Action

Toshio SAKAMOTO

Department of Physics and Atomic Energy Research Institute
College of Science and Technology, Nihon University, Tokyo 101

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In the method of stochastic quantization proposed by Parisi and Wu, the Langevin equation is set
by using an additional fictitious time. The results of the usual quantization are obtained by taking
the infinite limit of the time. For a system with a special class of potential, however, it is also
possible to formulate the stochastic quantization based on the Langevin equation associated with the
ordinary time. Applying this method, we investigate the stochastic quantization of the open-ends
Nambu-Gotô string. It is shown that the effective action derived out of the Langevin equation of the
string has the form of a supersymmetric string, though the fermionic degrees of freedom are ghost
variables.

§ 1. Introduction

The stochastic quantization method introduced by Parisi and Wu\(^1\) is known to
provide an alternative useful approach to the quantization of fields.\(^2\) The vacuum
expectation value of fields in the usual theory is obtained by taking an infinite
(fictitious) time limit in the stochastic expectation value of fields which is obtained as
the solution of the Langevin equations. Parisi and Sourlas\(^3\) also showed that in this
method, there arises a hidden supersymmetry in the measure of stochastic expecta-
tion;\(^4\) that is, we can naturally introduce supersymmetric (ghost) partners with fields
to define the measure. This means that the effective action in the stochastic method
can be expressed to be supersymmetric between the fields and their ghost partners,
which appear only in internal lines.

Now, in those formulation of the stochastic quantization, the Langevin equation
is set by using an additional fictitious time. In this paper, we show that the Langevin
equation can also be set by using the ordinary time for a system with a special class
of potentials having the form: \(U(q) = \Sigma (m/2)((\partial V/\partial q)^2 - (h/m)(\partial^2 V/\partial q^2))\).\(^5\) Then,
the result of the stochastic quantization is related directly to that of the usual
canonical quantization without taking the infinite time limit. This type of potentials
contains the potential of harmonic oscillator and, so, that of the relativistic string in
a specific gauge.

Applying this method, we also investigate the stochastic quantization of the
open-ends Nambu-Gotô string.\(^6\)

In the next section, the formulation of the ordinary time stochastic quantization
method is given.

In § 3, we investigate stochastic quantization of the Nambu-Gotô string in the
light-cone gauge and derive the Neveu-Schwarz like action as its effective action. In
such an effective action, the fermionic degrees of freedom appear as supersymmetric
(ghost) partners of string variables. Section 4 is devoted to a summary and discus-
sion. In the Appendix, a short review on the Neveu-Schwarz-Ramond string is given and the form of action used in § 3 is derived.\textsuperscript{7,8)

\section*{§ 2. Ordinary time stochastic quantization method}

In this section, we investigate an ordinary time stochastic quantization method for a system with \(N\) degrees of freedom characterized by the action

\[ S = \int ds \sum_{j=1}^{N} \left( \frac{m}{2} \left( \frac{dq^j(s)}{ds} \right)^2 - \left( \frac{\partial V(q)}{\partial q^j} \right)^2 + \hbar \left( \frac{\partial^2 V(q)}{\partial q^j(s)^2} \right) \right), \quad (m=\text{const}) \quad (2.1) \]

where \(V(q)\) is a function of \(q^2 = \sum q^i q^j\); the Hamiltonian operator\textsuperscript{*}) of this system becomes

\[ \hat{H} = \sum_{j=1}^{N} \left[ \hat{p}^j + \frac{m}{2} \left( \frac{\partial V}{\partial q^j} \right)^2 - \frac{\hbar}{m} \left( \frac{\partial^2 V}{\partial q^j(s)^2} \right) \right]. \quad (2.2) \]

It should be noticed that the lowest eigenvalue of \(\hat{H}\) is zero and the corresponding eigenstate, \(\hat{H}|0\rangle = 0\), has the form:

\[ \langle q|0\rangle = \langle 0|q \rangle \propto \exp \left[ -\frac{m}{\hbar} V(q) \right]. \quad (2.3) \]

According to the usual quantum mechanics, we can define the transition amplitude \(T(b, a)\) by

\[ T(b, a) = \langle q_b|\exp \left[ \frac{i}{\hbar} (t_b - t_a) \hat{H} \right]|q_a\rangle, \quad (2.4a) \]

\[ = e^{\left( \frac{m}{\hbar} \right)(V(q_b) - V(q_a))}\langle q_a|\exp \left[ -\frac{i}{\hbar} (t_b - t_a) \sum_{j=1}^{N} \left( \frac{1}{2m} \left( \hat{p}^j \right)^2 \right. \right. \]

\[ - \left. \left. \left. \frac{\hbar}{m} \left( \frac{\partial^2 V}{\partial q^j(s)^2} \right) \right) \right] \right] |q_a\rangle, \quad (2.4b) \]

where the similarity transformation \(\hat{H} = \exp((m/\hbar)\hat{V}))(\cdots)\exp(-\frac{m}{\hbar}\hat{V})\) has been carried out in the second equality of Eq. (2.4). Splitting up, here, the time interval into divisions of the length \(\varepsilon = (t_b - t_a)/M\), we have

\[ T(b, a) = \lim_{M \to \infty} e^{(m/\hbar)(V(q_b) - V(q_a))} \int_a^b \int \prod_{j=1}^{N} \prod_{k=1}^{M} dp_k d^2 q_k \delta^N(q_s^j - q_u^j) \]

\[ \times \exp \left[ -\frac{i}{\hbar} \sum_{j=1}^{N} \sum_{k=1}^{M} \frac{\varepsilon}{2m} \left( \hat{p}_k^j \right)^2 - 2m \hat{p}_k^j \left( \frac{q_k^j - q_{k-1}^j}{\varepsilon} \right) \right. \]

\[ + \left. \left. i \left( \frac{\partial V}{\partial q^j} \right)_k - \hbar \left( \frac{\partial^2 V}{\partial q^j(s)^2} \right)_k \right] \right], \quad (2.4c) \]

where \(q_0^j = q_a^j\) and \((\cdots)_k = (\cdots)_{q_{k-1} + q_{k-1}/2}\). Though, the similarity transformation in Eq. (2.4b) looks to spoil the convergence of the last path integral expression of \(T(b, a)\), after the Wick rotation \(\varepsilon \to -i\varepsilon\), the path integral representation becomes well convergent form:

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\textsuperscript{*}) Hereafter, operators are represented by the characters with "\(\wedge\)".
Ordinary Time Stochastic Quantization of Bosonic String

\[ T(b, a) = \lim_{\beta \to \infty} e^{(m/\hbar)\{(\varphi_b^{\prime}) - (\varphi_0^{\prime})\}} \int_a^b \int \left[ \frac{\sum_{k=1}^{M} dp_k dq_k}{2\pi\hbar} \right] \delta^N(q_o - q_{o'}) \]

\[ \times \exp \left[ -\frac{1}{\hbar} \sum_{j=1}^{M} \sum_{k=1}^{N} \frac{1}{2m} \left( \frac{\partial^2 V}{\partial q_j^{\prime}} - 2im \left( \frac{\partial^2 V}{\partial q_j^{\prime^2}} \right)_k \right) \right], \]  

(2.5)

to which we can carry out the integration with respect to the \( p_k' \)'s explicitly and get

\[ T(b, a) = \lim_{\beta \to \infty} e^{(m/\hbar)\{(\varphi_b^{\prime}) - (\varphi_0^{\prime})\}} \int_a^b \prod_{j=1}^{N} \frac{m}{2\pi\hbar} dq_j' \]

\[ \times \delta^N(q_o' - q_{o'}) \exp \left[ -\frac{1}{\hbar} \sum_{j=1}^{M} \sum_{k=1}^{N} \frac{1}{2m} \left( \frac{\partial^2 V}{\partial q_j^{\prime}} \right)_k \right], \]

(2.6)

If we introduce the variables \( W_k' \) defined by

\[ W_k' = \frac{q_k' - q_{k-1}'}{\epsilon} + \left( \frac{\partial V}{\partial q_j'} \right)_k, \]  

(2.7)

the right-hand side of Eq. (2.6) can be expressed, further, in the following form:

\[ T(b, a) = e^{(m/\hbar)\{(\varphi_b^{\prime}) - (\varphi_0^{\prime})\}} P(b, a) , \]  

(2.8a)

where

\[ P(b, a) = \lim_{\beta \to \infty} \int_a^b \int \left[ \frac{\prod_{j=1}^{N} \prod_{k=1}^{M} \frac{m}{2\pi\hbar} dW_k'}{\delta^N(q_o' - q_{o'})} \right] \delta^N(q_o' - q_{o'}) \det \left( \frac{\partial W_k'}{\partial q_j'} \right) \]

\[ \times \prod_{k=1}^{M} \delta^N \left[ W_k' - \frac{q_k' - q_{k-1}'}{\epsilon} - \left( \frac{\partial V}{\partial q_j'} \right)_k \right] \]

\[ \times \exp \left[ -\frac{1}{\hbar} \sum_{j=1}^{N} \sum_{k=1}^{M} \frac{1}{2m} \left( W_j' \right)^2 \right] , \]

(2.8b)

Now, Eqs. (2.7)~(2.8b) give us another approach to quantization of the system described by the action (2.1); that is, the stochastic quantization method which regards Eq. (2.7) as the Langevin equation with Gaussian white noises \( W_k' \) normalized so that

\[ \langle W_k' \rangle_w = 0 , \quad \langle W_k' W_k' \rangle_w = \frac{\hbar}{m} \delta_{kk'} \frac{\delta_{kk'}}{\epsilon} , \]

(2.9a)

\[ \text{(*)} \]

\[ \det \left( \frac{\partial W_k'}{\partial q_j'} \right) = \exp \left[ \text{Tr} \log \left( \frac{\partial W_k'}{\partial q_j'} \right) \right] \]

\[ = \exp \left[ \text{Tr} \log \left[ \frac{1}{\epsilon} \left( \delta_{0}(\delta_{k}^{\prime} - \delta_{k-1}^{\prime}) + \epsilon (\delta_{k}^{\prime} + \delta_{k-1}^{\prime}) \left( \frac{\partial^2 V}{\partial q_j^{\prime}} \right)_k \right) \right] \right] \]

\[ = \lim_{\hbar \to 0} \left[ \prod_{j=1}^{N} \frac{1}{\epsilon} \right] \exp \left[ \frac{1}{\hbar} \sum_{j=1}^{N} \sum_{k=1}^{M} \epsilon \hbar \left( \frac{\partial^2 V}{\partial q_j^{\prime}} \right)_k \right] . \]
where \( \langle \cdots \rangle_w \) denotes the average of \( \cdots \) over \( W \) distribution. If it is necessary, one can write this average explicitly as

\[
\langle \cdots \rangle_w = \lim_{M \to \infty} \int_a^b \left[ \prod_{j=1}^N \prod_{k=1}^M \sqrt{\frac{EM}{2\pi h}} dW_k \right] \langle \cdots \rangle \exp \left[ -\frac{1}{h} \sum_{j=1}^N \sum_{k=1}^M \frac{EM}{2} (W_k^2) \right],
\]

\( \langle 1 \rangle_w = 1 \{ 2 \cdot 9 \text{b} \}

In terms of random variables \( q'_w(b, a) \), the solution of Eq. (2・7) with the initial condition \( q'_w(a, a) = q'_a \), the probability density \( P(b, a) \) can be expressed as follows:\(^9,\text{10}\)

\[
P(b, a) = \langle \delta^N [q_b^j - q'_w(b, a)] \rangle_w .
\]

Therefore, from the viewpoint of the stochastic process governed by the Langevin equation (2・7), the transition amplitude (2・8a) becomes

\[
T(b, a) = e^{(m/h)(V(q_b) - V(q_a))} \langle \delta^N [q_b^j - q'_w(b, a)] \rangle_w .
\]

Now by inverting the process from Eq. (2・1) to (2・11), we can get the effective action of the system governed by Eqs. (2・7) and (2・9a). In that case, we can express the functional determinant \( \det(\partial W/\partial q) \) in Eq. (2・8b) by introducing fermionic ghost variables and then, we can derive an effective action which has a form different from Eq. (2・1).\(^11,\text{14}\)

To see this, let us write the functional determinant as

\[
\det \left( \frac{\partial W_{k,j}}{\partial q_{l,i}} \right) = \det \left[ \frac{1}{\epsilon} \left( \delta_{ij} (\delta_{k,j} - \delta_{k,j-1}) + \epsilon \left( \frac{\delta^2 V}{2} \right)_{kl} \right) \right],
\]

\( \{ 2 \cdot 12 \text{a} \}

\[
= \lim_{M \to \infty} \left[ \prod_{j=1}^N \prod_{k=1}^M \frac{1}{\epsilon} \right] \int_a^b \left[ \prod_{j=1}^N \prod_{k=1}^M dA_k^* dA_k \right] \exp \left[ -\frac{1}{h} S_{\text{ghost}} \right],
\]

\( \{ 2 \cdot 12 \text{b} \}

where \( A_k^j \) and \( A_k^{*-j} \) are Grassmann variables and \( S_{\text{ghost}} \) is

\[
S_{\text{ghost}} = \frac{h}{2} \sum_{k=1}^M \sum_{j=1}^N A_k^j \left\{ \delta_{ij} \left( \frac{A_k^j - A_{k-1}^j}{\epsilon} \right) + \left( \frac{\delta^2 V}{2} \right)_{kl} \left( \frac{A_k^j + A_{k-1}^j}{2} \right) \right\},
\]

\( \{ 2 \cdot 12 \text{c} \)

where \( A_0 = 0 \). Then one can define the effective action of the system constructed out of the \( q^j \)'s and fermionic ghosts \( (A, A^*) \) by

\[
\bar{T}(b, a) = \lim_{M \to \infty} \int_a^b \left[ \prod_{j=1}^N \prod_{k=1}^M \sqrt{\frac{m}{2\pi h \epsilon}} dq_k^j dq_k^{*-j} dA_k^* dA_k \right]

\times \exp \left[ -\frac{1}{h} S_{\text{eff}}(b, a) \right].
\]

\( \{ 2 \cdot 13 \}

If we, here, noticed that

\[
V(q_b) - V(q_a) \approx \lim_{M \to \infty} \sum_{j=1}^N \sum_{k=1}^M \frac{q_k^j - q_{k-1}^j}{\epsilon} \left( \frac{\partial V}{\partial q^j} \right)_k,
\]

the effective action can be obtained in the following form.\(^15\)
It should be stressed that $S_{\text{ghost}}$ is nothing but the stochastic description of the potential term:

$$-\hbar \sum_{k=1}^{M} \varepsilon \sum_{j=1}^{N} \frac{1}{2} \left( \frac{\partial^2 V}{\partial q_j^2} \right)^k,$$

in the classical action (2.1).

Finally, let us discuss the method calculating the vacuum expectation value of the time ordered product of the $\hat{q}^i$s by means of the stochastic average over $\hat{W}$ distribution.

According to the usual quantum mechanics, in view of $\hat{H}|0\rangle=0$, we can get easily

$$\langle 0| T[\hat{q}(s_1)\hat{q}(s_2)\cdots\hat{q}(s_n)]|0\rangle$$

$$= \lim_{t_b, t_a \to -\infty} \int \int (\prod_{k=1}^{n} dq_k d\hat{q}_k) \langle 0| q_b \langle q_a |0\rangle \int \int \int \prod_{k=1}^{n} dq_k(s_k)$$

$$\times T[\bar{T}(b, s_1)q'(s_1)\bar{T}(s_2)q'(s_2)\bar{T}(s_3)q'(s_3)\cdots q'(s_n)\bar{T}(s_n, a)]$$

provided that $t_b > s_1, s_2, \ldots, s_n > t_a$. Combining (2.11) and (2.16), we can write the vacuum expectation value of the time ordered product in such a form as

$$\langle 0| T[\hat{q}(s_1)\hat{q}(s_2)\cdots\hat{q}(s_n)]|0\rangle$$

$$= \lim_{t_b, t_a \to -\infty} \int \int (\prod_{k=1}^{n} dq_k d\hat{q}_k) \langle q_a |0\rangle^2 \int \int \int \prod_{k=1}^{n} dq_k(s_k)$$

$$\times T[\delta^N[q_b - q_w(b, s_1)]\delta^N[q(s_1) - q_w(s_1, s_2)]\delta^N[q(s_2) - q_w(s_2, s_3)]\cdots q^m(s_n)\delta^N[q(s_n) - q_w(s_n, a)]|w\rangle$$

$$= \lim_{t_b, t_a \to -\infty} \int (\prod_{k=1}^{n} dq_k) \rho_0[q_a] \langle q_w(s_1, a)q_w(s_2, a)\cdots q_w(s_n, a)|w\rangle$$

$$\left( \rho_0[q_a]=\langle q_a |0\rangle^2 \propto \exp \left[ -\frac{2m}{\hbar} V(q_a) \right] \right).$$

In the second equality of Eq. (2.17), we have used the relation such as $q^i(s)\delta^N[q(s) - q_w(s, s')]=q^i(s, s')\delta^N[q(s) - q_w(s, s')]$ and $\langle A_w \rangle_{w=qB_w}=\langle A_w B_w \rangle_w$ ($A_w$ and $B_w$ are stochastic variables defined in non-overlapping time intervals).

Equation (2.17b) is the expression to the vacuum expectation value in our stochastic formulation.

§ 3. The stochastic description of open-ends

Nambu-Gotô string with fermionic ghost

As an application of the method developed in the previous section, we here investigate the stochastic quantization of open-ends Nambu-Gotô string and try to find the effective action for the fermionic ghosts. For this purpose, we have to
start with a gauge fixed action of the string, since it is necessary to set up a Langevin equation based on the string's equation of motion without constraints. As a simplest way, let us put the orthonormal gauge condition; then the action for Nambu-Goto string has the following form:

\[ S = -\left(\frac{\kappa}{2}\right) \int d^2x \sum_{\alpha=0}^{d-1} \left[ (\partial_\alpha x^\alpha(z))^2 - (\partial_\alpha x^\alpha(z))^2 \right] , \tag{3.1} \]

where \( x^\alpha(z) \) is the position variable of the string. \( z^0 \) and \( z^1 \) are time-like and space-like parameter of the string, respectively. In this gauge, there still remain gauge degrees of freedom which can be, however, eliminated by putting additional light-cone* gauge conditions:

\[ x^+(z^0, z^1) = X^+(z^0) , \quad \partial_0 X^+(z^0) = 1 , \tag{3.2} \]

where we have divided \( x^\alpha(z) \) into its center of mass part \( X^\alpha(z^0) \) and relative coordinate \( \bar{x}^\alpha(z) \); that is, \( x^\alpha(z) = X^\alpha(z^0) + \bar{x}^\alpha(z) \). By assuming the boundary condition of open-ends string with \( 0 \leq z^1 \leq \pi \), \( x^\alpha(z) \) is expanded in the fourier series of \( \cos(nz^1) \):

\[ x^\alpha(z) = X^\alpha(z^0) + \sqrt{\frac{2}{\pi}} \sum_{\ell = 0}^{\infty} \bar{x}^\alpha_\ell(z_\ell^0) \cos(nz^1) . \tag{3.3} \]

In virtue of Eqs. (3.2) and (3.3), (3.1) is reduced to

\[ S = -\left(\frac{\pi \kappa}{2}\right) \int d^2x \sum_{\ell = 0}^{d-2} \sum_{n = 1}^{\infty} \left[ (\partial_\ell \bar{x}^\ell_{n}(z_\ell^0))^2 \right. \left. - (\bar{x}^\ell_{n}(z_\ell^0))^2 \right] + S_{\text{rel}} , \tag{3.4a} \]

where

\[ S_{\text{rel}} = \left(\frac{\kappa}{2}\right) \int d^2x \sum_{\ell = 1}^{d-2} \sum_{n = 1}^{\infty} \left[ (\partial_\ell \bar{x}^\ell_{n}(z_\ell^0))^2 \right. \left. - (\bar{x}^\ell_{n}(z_\ell^0))^2 \right] . \tag{3.4b} \]

Carrying out, here, the Wick rotation \( z^0 \rightarrow -iz^0 \), \( x^+ \rightarrow -ix^0 \), \( \bar{x}^\ell_{n}(z^0) \rightarrow \bar{x}^\ell_{n}(z_\ell^0) \), and putting \( \bar{S} = -iS \), \( \bar{S}_{\text{rel}} = -iS_{\text{rel}} \), we have

\[ \bar{S} = \left(\frac{\pi \kappa}{2}\right) \int d^2x \sum_{\ell = 1}^{d-2} \sum_{n = 1}^{\infty} \left[ (\partial_\ell \bar{x}^\ell_{n}(z_\ell^0))^2 \right. \left. + (\bar{x}^\ell_{n}(z_\ell^0))^2 \right] + \bar{S}_{\text{rel}} , \tag{3.5a} \]

where

\[ \bar{S}_{\text{rel}} = \left(\frac{\kappa}{2}\right) \int d^2x \sum_{\ell = 1}^{d-2} \sum_{n = 1}^{\infty} \left[ (\partial_\ell \bar{x}^\ell_{n}(z_\ell^0))^2 + (\bar{x}^\ell_{n}(z_\ell^0))^2 \right] . \tag{3.5b} \]

Since we are mainly interested in the supersymmetric structure between the relative coordinates of the string and their fermionic ghosts, we hereafter confine our attention to the action (3.5b).**

Each term in the summation of \( \bar{S}_{\text{rel}} \) has the form, to which one can apply the method in the previous section; that is, \( \bar{x}^\ell_{n} \), \( \kappa \) and \( \Sigma_{\ell = 2}^{d-2} \Sigma_{n = 1}^{\infty} (n/2)(\bar{x}^\ell_{n})^2 \) in Eq. (3.5b) are corresponding to \( q^\ell, m, V(q) \), respectively. In the case of string, however, the last

* The light-cone components of a vector \( (A^\alpha) \), \( \alpha = 0, 1, \ldots, d-1 \) is defined by \( A^\pm = (1/\sqrt{2})(A^\alpha \pm A^{\alpha \, -}) \). We also call \( A^\ell, (\ell = 1, \ldots, d-2) \) as the transverse components of \( (A^\alpha) \).

** The treatment of the center of mass part is discussed in §4.
Ordinary Time Stochastic Quantization of Bosonic String

429
term on the r.h.s. of Eq. (2.1) gives rise to a c-number infinity and so, it is convenient to start with the modified action

\[ S_{st} = S_{tr} - \sum_{n=1}^{\infty} n^s \left( \frac{d-2}{2} \int dx^n \right) \]  

(3.6)
rather than the action (3.5b). Then the Langevin equations, which are characteristic of the above string in the stochastic quantization, become

\[ W_n'(z^0) = (\partial_0 + n) \bar{x}'(z^0) \]  

(3.7)
where \( W_n'(z^0) \) are white noises defined by

\[ \langle \cdots \rangle_w = \int [dw] \langle \cdots \rangle \exp \left[ -\left( \frac{\kappa}{2} \right) \sum_{j=1}^{d-2} \sum_{n=1}^{\infty} W_{n'}(z^0) \right] \]  

(3.8a)
with the normalization

\[ \langle 1 \rangle_w = 1 \]  

(3.8b)

Now, let us write the solution of Eq. (3.7) under the initial condition \( \bar{x}'(z^0_0) = \bar{x}'(z^0, z^0_0) \), since it is a functional of \( W_n'(z^0) \); we also write the average of a functional \( F[\bar{x}'] \) of \( \bar{x}' \) over the \( W \) as \( \langle F[\bar{x}'] \rangle_w \). Then, by considering Eqs. (2.15a) and (2.15b), the expectation value of \( F[\bar{x}'] \) by the string's ground state in the usual canonical quantization can be related to the stochastic average \( \langle F[\bar{x}'] \rangle \) by

\[ \langle F[\bar{x}'] \rangle_{\rho_0} = \int \left( \prod_{j=1}^{d-2} \prod_{n=1}^{\infty} d\bar{x}'(z^0) \right) \langle F[\bar{x}'] \rangle_w \rho_0 \left[ \bar{x}' \right] , \]  

(3.9a)

\[ = \left( \int [d\bar{x}] \det \left( \frac{\partial W}{\partial \bar{x}} \right) \prod_{n=1}^{\infty} \delta^{d-2} \left[ W_n'(z^0) - (\partial_0 + n) \bar{x}'(z^0) \right] \right) \]  

\[ \times \int \left( \prod_{j=1}^{d-2} \prod_{n=1}^{\infty} d\bar{x}'(z^0) \right) F[\bar{x}] \rho_0 \left[ \bar{x}' \right] , \]  

(3.9b)

\[ = \int \int [d\bar{x} dA^* dA] F[\bar{x}] \langle 0 | \bar{x} b \rangle \]  

\[ \times \exp \left[ -\int d\bar{x}' \sum_{j=1}^{d-2} \sum_{n=1}^{\infty} \left( \frac{\kappa}{2} \right) ((\partial_0 \bar{x}'(z^0))^2 + (n \bar{x}'(z^0))^2) \right. \]  

\[ + A_n^*(z^0)(\partial_0 + n) A_n(z^0) \right] \langle \bar{x} | 0 \rangle , \]  

(3.9c)

\[ = \langle 0 | T[F[\bar{x}]] 0 \rangle , \]  

(3.9d)

where

*) By using the \( \zeta \)-function regularization method, the second term on the r.h.s. of Eq. (3.6) gives rise to the well-known anomaly

\[ \lim_{s \to -1} s \prod_{n=1}^{\infty} n^{-s} = \frac{d-2}{2} \lim_{s \to -1} \zeta(s) = -\frac{d-2}{24} . \]

**) We hereafter alter the notation in the stochastic method from splitting-time-expression to continuous-time-expression.
\begin{align}
\langle 0 | \bar{x}_a \rangle &= \langle \bar{x}_a | 0 \rangle \\
&= \left[ \prod_{j=1}^{d-2} \prod_{n=1}^{\infty} \sqrt{\frac{\omega n}{\pi}} \right] \exp \left[ - \sum_{j=1}^{d-2} \sum_{n=1}^{\infty} \left( \frac{\omega n}{2} \right) (\bar{x}_n^j a)^2 \right] \\
\text{(3.9e)}
\end{align}

and

\begin{align}
\rho_0[\bar{x}_n^j a] &= \langle 0 | \bar{x}_a \rangle \langle \bar{x}_a | 0 \rangle, \\
&= \left[ \prod_{j=1}^{d-2} \prod_{n=1}^{\infty} \sqrt{\frac{\omega n}{\pi}} \right] \exp \left[ - \sum_{j=1}^{d-2} \sum_{n=1}^{\infty} \omega n (\bar{x}_n^j a)^2 \right] \\
\text{(3.9f)}
\end{align}

is the ground state probability density at initial time, and \( A_n^i(\bar{z}^0), A_n^i(\bar{z}^0) \) are Grassmann variables. In Eq. (2.9c), we have used the well-known property of the integration with respect to fermionic variables. Carrying out, here, the (inverse) Wick rotation \( z^0 \rightarrow i\bar{z}^0 \), \( \langle F[\bar{x}_a] \rangle_{\rho_0} \) can be rewritten as

\begin{align}
\langle F[\bar{x}_a] \rangle_{\rho_0} &= \int \int \int [d\bar{x}][dA^*][dA] F[\bar{x}] \exp [iS_{\text{eff}}] \langle 0 | \bar{x}_a \rangle \langle \bar{x}_a | 0 \rangle, \\
\text{(3.10a)}
\end{align}

where \( S_{\text{eff}} \) is the effective action in the real time:

\begin{align}
S_{\text{eff}} &= \int d\bar{z}^0 \sum_{j=1}^{d-2} \sum_{n=1}^{\infty} \left( \frac{\omega n}{2} \right) \left( (\partial_0 \bar{x}_n^j(\bar{z}^0))^2 - (n \bar{x}_n^j(\bar{z}^0))^2 \right) \\
&\quad + i \left( A_n^i(\bar{z}^0)(\partial_0 + i\nu) A_n^i(\bar{z}^0) \right). \\
\text{(3.10b)}
\end{align}

Further, under the scale transformations

\begin{align}
\Gamma_N^i(\bar{z}^0) &= e^{-iz^0/2} A_N^i(\bar{z}^0), \\
\Gamma_N^j(\bar{z}^0) &= e^{iz^0} A_N^j(\bar{z}^0), \\
\text{(3.11)}
\end{align}

the effective action (3.10b) is transformed into

\begin{align}
S_{N,s} &= \int d\bar{z}^0 \sum_{j=1}^{d-2} \sum_{n=1}^{\infty} \left( \frac{\omega n}{2} \right) \left( (\partial_0 \bar{x}_n^j(\bar{z}^0))^2 - (n \bar{x}_n^j(\bar{z}^0))^2 \right) \\
&\quad + i \sum_{N=1/2}^{\infty} \left[ \Gamma_N^i(\bar{z}^0)(\partial_0 + iN) \Gamma_N^j(\bar{z}^0) \right], \quad \left( N = n - \frac{1}{2} \right) \\
\text{(3.12)}
\end{align}

which shows the relation between the effective action in the stochastic quantization of the bosonic string and the action of the Neveu-Schwarz string. Therefore, we can get the following expression of \( \langle F[\bar{x}_a] \rangle_{\rho_0} \):

\begin{align}
\langle F[\bar{x}_a] \rangle_{\rho_0} &= \int \int \int [d\bar{x}][dA^*][dA] F[\bar{x}] \exp [iS_{N,s}] \langle 0 | \bar{x}_a \rangle \langle \bar{x}_a | 0 \rangle, \\
\text{(3.13)}
\end{align}

where \( S_{N,s} \) is given by (3.12) and use has been made of the relation \([d\bar{x}][dA^*][dA] = [d\bar{x}][dA^*][dA] \] = [d\bar{x}][dA^*][dA]]).

We note that though the effective action (3.12) has the same form as the Neveu-Schwarz string action (A.8b) with respect to relative coordinates, the fermionic variables \( \Gamma_N^i(\bar{z}^0) \) and \( \Gamma_N^j(\bar{z}^0) \) still remain as the unobservables like the F. P. ghosts in gauge field theories. In this section, we have discussed the free string only and the interaction can be treated, according to our method, if it can be introduced by
the substitution \((n\vec{x}_n')^2 \to [(n\vec{x}_n' + (\partial V(\vec{x})/\partial \vec{x}_n'))^2 - (n + (\partial^2 V(\vec{x})/\partial \vec{x}_n'^2)]\) in the action (3.4b).

§ 4. Discussion and concluding remarks

The results in §§ 2 and 3 are summarized as follows:

In § 2, we have discussed the stochastic quantization of the system described by the action (2.1) and shown that the usual result of canonical quantization is obtained based on the ordinary time Langevin equation (2.7) associated with the white noise (2.9a). For example, the vacuum expectation values of the time ordered product of canonical variables could be expressed in terms of stochastic variables as (2.17b). It was also found that such a stochastic method naturally gives rise to an effective action of the system, which is equivalent to (2.1) and has a supersymmetry between the ordinary variables and fermionic ghost ones.

In § 3, we have applied our stochastic method to the quantization of the open-ends Nambu-Gotô string and derived the action of some kinds of supersymmetric string as its effective one. As the remaining important points, we shall comment on the following:

First, the form of effective action derived out of the Langevin equation (2.7) varies depending on the definition of the potential gradient \((\partial V/\partial q)_k\) in the discrete time. For instance, if we write the Langevin equation in such a form as

\[
W_k^j = \frac{q_k^j - q_{k-1}^j}{\epsilon} + \left(\frac{\partial V}{\partial q_{k-1}^j}\right),
\]

then the form of effective action becomes (2.1), the action without fermionic ghosts, due to the trivial form of determinant:

\[
\det\left(\frac{\partial W_i^j}{\partial q_{i-1}^j}\right) = \det\left(\frac{1}{\epsilon} \delta_{i-1}^j \delta_{i-1}^j - \delta_{i-1}^j \delta_{i-1}^j\right) = \lim_{N \to 0} \prod_{i=1}^{N} \frac{1}{\epsilon}.
\]

Secondly, let us consider the center of mass part of the string, which has been disregarded simply in our stochastic treatment of the open-ends Nambu-Gotô string. The Langevin equation for the center of mass variable \(X_i(z_0)\) is given by

\[
W_0^i(z_0) = \frac{\partial}{\partial X_i(z_0)}
\]

with the definition of the white noise \(W_0\) normalized so that

\[
\langle W_0^i(z_0) W_0^j(z_0') \rangle_{\omega_0} = (\pi \epsilon)^{-1} \delta_{ij} \delta(z_0 - z_0').
\]

Then, one can derive the effective action for the center of mass part in the following form:

\[
\overline{S}_{\text{L}=c} = - \int dz^2 \left[ \pi \epsilon \partial_{\xi} X^{-\prime}(z_0') \right] + \int dz^2 \left[ \frac{\pi \epsilon}{2} \sum_{i=1}^{d-2} \left( \partial_{\xi} X_i(z_0') \right)^2 \right],
\]

except the total derivative term. If we take the center of mass part into account, then the Langevin equations in Eqs. (3.7) and (4.3) can be unified into a Langevin equation of string's density variables \(x_i(z_0, z_1)\) \((0 \leq z_1 \leq \pi)\) such that
\[ W^i(z^0, z^1) = \partial_0 x^i(z^0, z^1) - \int_0^\pi dz^0' K(z^1, z^0') \partial_1 x^i(z^0, z^1). \] (4.6a)

Here \( K(z^1, z^0') \) is the integration kernel\(^{18}\)

\[ K(z^1, z^0') = \frac{1}{\pi} \frac{\sin z^0'}{\cos z^1 - \cos z^0'}. \] (4.6b)

and the white noise in the density form \( W^i(z^0, z^1) \) is defined by

\[ W^i(z^0, z^1) = W_{0i}^i(z^0) + \sqrt{\frac{2}{\pi}} \sum_{j=1}^{\infty} W_{ji}^i(z^0) \cos(nz^1). \] (4.7)

Finally, we note the presence of similarity between the effective action (3.12) and that of the Neveu-Schwarz string in spite of the fact that \( \Gamma_i^j \) and \( \Gamma_i^j^* \) are ghosts. In other words, the real form of Green functions for the Neveu-Schwarz string can be derived from the generating function defined out of the effective action (3.12) with external sources for \( \Gamma_i^j \) and \( \Gamma_i^j^* \). Then, it is interesting that such an effective action with external sources, again, can be obtained from the Langevin equation (3.7) with a white noise depending on the external sources. These facts may be useful when we consider the bosonization of spinning string.

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**Appendix**

— Neveu-Schwarz-Ramond String —

Under the orthonormal gauge condition, the action of the Neveu-Schwarz-Ramond string has the following form:\(^{4,10}\)

\[ S = -\int dz^0 \int dz \left[ \frac{K}{2} \partial_\alpha x'^a \partial_\alpha x_a + i \frac{1}{2} \overline{\psi}^a \gamma^\alpha \partial_\alpha \psi_a \right]. \] (A.1)

Here, the \( \phi^a \) fields are supposed to satisfy the Majorana condition

\[ \phi^a = -\gamma^5 \phi^{a*}, \quad (\ast: \text{complex conjugate}) \] (A.2)

and the boundary conditions for \( x'^a(z^0, z^1) \) and \( \phi^a(z^0, z^1) \) are

\[ \partial_0 x'^a(z^0, z^1)|_{z^1=0, \pi} = 0, \] (A.3a)

\[ \phi^a_0(z^0, 0) = -i \phi^a(z^0, 0), \] (A.3b)

\[^*\] In our notation, \( \gamma \)-matrices in two dimensions are

\[ \gamma^0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \gamma^1 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad \gamma^5 = \gamma^0 \gamma^1 = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \]

and the adjoint \( \tilde{\phi} \) of \( \phi \) is defined by \( \tilde{\phi} \equiv \phi^\dagger \gamma^0 \), where \( \phi^\dagger \) is hermitian conjugate of \( \phi \).
Ordinary Time Stochastic Quantization of Bosonic String

\[ \phi_0^a(z^0, \pi) = \pm i \phi_1^a(z^0, \pi) . \]  
(A·3c)

\( \phi_0^a, \phi_1^a \) are the components of \( \phi^a \) and the plus (minus) sign in Eq. (A·3c) corresponds to the boundary condition of the Neveu-Schwarz (Ramond) string.

By using the fields \( \Gamma^a, Q^a \) defined in the extended domain of \( z'(-\pi < z' < \pi) \):

\[ \Gamma^a(z^0, z^1) = \phi_0^a(z^0, z^1) \theta(z^1) - i \phi_1^a(z^0, -z^1) \theta(-z^1), \]  
(A·4a)

\[ Q^a(z^0, z^1) = x^a(z^0, z^1) \theta(z^1) + x^a(z^0, -z^1) \theta(-z^1), \]  
(A·4b)

the action (A·1) can also be written as

\[ S = -\int dz^0 \int dz \left[ \frac{k}{4} \{ \partial_0 Q^a \partial_0 Q_a - \partial_1 Q^a \partial_1 Q_a \} + \frac{i}{2} (\partial_0 - \partial_1) \Gamma_a \right] . \]  
(A·5)

In order to remove the remaining gauge degrees of freedom from the action (A·5), we further put the following light-cone gauge conditions:

\[ Q^+(z^0, z^1) = X^+(z^0), \quad \partial_0 X^+(z^0) = 1, \quad \Gamma^+(z^0, z^1) = 0, \]  
(A·6)

where the superscript "+" means light-cone variables in the external Minkowski space-time.

Now, under the boundary condition (A·3), \( Q^a(z), \Gamma^a(z) \) are expanded in the following fourier series:

\[ Q^a(z^0, z^1) = X^a(z^0) + \sqrt{\frac{2}{\pi}} \sum_{n=1}^{\infty} \bar{x}_n^a(z^0) \cos(nz^1) \]  
(A·7a)

and

\[ \Gamma^a(z^0, z^1) = \sqrt{\frac{1}{2\pi}} \sum_{n=-\infty}^{\infty} \Gamma_n^a(z^0) e^{-in\pi z^1}, \]  
(A·7b)

where \( N \) is a half-integer (integer) in the case of the Neveu-Schwarz (Ramond) string. It should be noted that the zero mode \( \Gamma_0^a(z^0) \) of \( \Gamma^a(z^0, z^1) \) does not exist for the Neveu-Schwarz string in Eq. (A·7b).

Substituting Eqs. (A·6), (A·7a) and (A·7b) into the action (A·5), the action of the Neveu-Schwarz-Ramond string, in the light-cone gauge, becomes

\[ S = -\int dz^0 \left[ \frac{\pi k}{2} \{ \partial_0 X^-(z^0) - \sum_{j=1}^{d-2} (\partial_0 X^j(z^0))^2 \} - \sum_{j=1}^{d-2} \frac{i}{2} \Gamma_j(z^0) \partial_0 \Gamma_j(z^0) \right] + S_{\text{rel}}, \]  
(A·8a)

where

\[ S_{\text{rel}} = \int dz^0 \sum_{j=1}^{d-2} \left[ \frac{k}{2} \sum_{n=1}^{\infty} \{ (\partial_0 \bar{x}_n^j(z^0))^2 - (nx_{-n}^j(z^0))^2 \} \right] \]

\[ + i \sum_{n > 0} (\Gamma_n(z^0)(\partial_0 - iN) \Gamma_{-n}(z^0) + \frac{1}{2} \partial_0 (\Gamma_n(z^0) \Gamma_{-n}(z^0))) \]

\[ (\Gamma_n(z^0) \) is equal to \( \Gamma_n(z^0) \) \]  
(A·8b)
References

14) After completing this work, the author also found the following: H. Ezawa and J. R. Klauder,